Nonlinear Realization of the Local Conform-Affine Symmetry Group for Gravity in the Composite Fiber Bundle Formalism

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A gauge theory of gravity based on a nonlinear realization (NLR) of the local Conform-Affine (CA) group of symmetry transformations is presented. The coframe fields and gauge connections of the theory are obtained. The tetrads and Lorentz group metric are used to induce a spacetime metric. The inhomogenously transforming (under the Lorentz group) connection coefficients serve as gravitational gauge potentials used to define covariant derivatives accommodating minimal coupling of matter and gauge fields. On the other hand, the tensor valued connection forms serve as auxillary dynamical fields associated with the dilation, special conformal and deformational (shear) degrees of freedom inherent in the bundle manifold. The bundle curvature of the theory is determined. Boundary topological invariants are constructed. They serve as a prototype (source free) gravitational Lagrangian. The Bianchi identities, covariant field equations and gauge currents are obtained.

Key Words: gauge symmetry, conform-affine Lie algebra, gravity, fiber bundle formalism.

I. INTRODUCTION

Quantum theory and relativity theory are two fundamental theories in modern physics. The so-called standard model is currently the most successful relativistic quantum theory in particle physics. It is a non-Abelian gauge theory (Yang-Mills theory) associated with the internal symmetry group $SU(3) \times SU(2) \times U(1)$, in which the $SU(3)$ color symmetry for the strong force in quantum chromodynamics is treated as exact whereas the $SU(2) \times U(1)$ symmetry responsible for generating the electro-weak gauge fields is spontaneously broken. So far as we know, there are four fundamental forces in Nature; namely, electromagnetic force, weak force, strong force and gravitational force. The standard model covers the first three, but not the gravitational interaction. In general relativity, the geometrized gravitational field is described by the metric tensor $g_{\mu\nu}$ of pseudo-Riemannian spacetime, and the field equations that the metric tensor satisfies are nonlinear. This nonlinearity is indeed a source of difficulty in quantization of general relativity. Since the successful standard model in particle physics is a gauge theory in which all the fields mediating the interactions are represented by gauge potentials, a question arises as to why the fields mediating the gravitational interaction are different from those of other fundamental forces. It is reasonable to expect that there may be a gauge theory in which the gravitational fields stand on the same footing as those of other fields. This expectation has prompted a re-examination of general relativity from the gauge theoretical point of view.

While the gauge groups involved in the standard model are all internal symmetry groups, the gauge groups in general relativity must be associated with external spacetime symmetries. Therefore, the gauge theory of gravity will not be a usual Yang-Mills theory. It must be one in which gauge objects are not only the gauge potentials but also tetrads that relate the symmetry group to the external spacetime. For this reason we have to consider a more complex nonlinear gauge theory. In general relativity, Einstein took the spacetime metric as the basic variable representing gravity, whereas Ashtekar employed the tetrad fields and the connection forms as the fundamental variables. We also consider the tetrads and the connection forms as the fundamental fields.

R. Utiyama (1956) was the first to suggest that gravitation may be viewed as a gauge theory [1] in analogy to the Yang-Mills [2] theory (1954). He identified the gauge potential due to the Lorentz group with the symmetric connection of Riemann geometry, and constructed Einstein's general relativity as a gauge theory of the Lorentz group $SO(3,\, 1)$ with the help of tetrad fields introduced in an *ad hoc* manner. Although the tetrads were necessary components of the theory to relate the Lorentz group adopted as an internal gauge group to the external spacetime, they were not introduced as gauge fields. In 1961, T.W.B. Kibble [3] constructed a gauge theory based on the Poincaré group $P(3,\, 1) = T(3,\, 1) \rtimes SO(3,\, 1)$ ($\rtimes$ represents the semi-direct product) which resulted in the Einstein-Cartan theory characterized by curvature and torsion. The translation group $T(3,\, 1)$ is considered responsible for generating the tetrads as gauge fields. Cartan [4] generalized the Riemann geometry to include torsion in addition to curvature. The torsion (tensor) arises from an asymmetric connection. D.W. Sciama [5], and others (R. Fikelstein [6], Hehl [7, 8]) pointed out that intrinsic spin may be the source of torsion of the underlying spacetime manifold.

Since the form and role of the tetrad fields are very different from those of gauge potentials, it has been thought that even Kibble's attempt is not satisfactory as a full gauge theory. There have been a number of gauge theories

of gravitation based on a variety of Lie groups [7, 8, 9, 10, 11, 12, 13]. It was argued that a gauge theory of gravitation corresponding to general relativity can be constructed with the translation group alone in the so-called teleparallel scheme. Inomata *et al.* [14] proposed that Kibble's gauge theory could be obtained in a manner closer to the Yang-Mills approach by considering the de Sitter group $SO(4,\,1)$ which is reducible to the Poincaré group by group-contraction. Unlike the Poincaré group, the de Sitter group is homogeneous and the associated gauge fields are all of gauge potential type. By the Wigner-Inönu group contraction procedure, one of five vector potentials reduces to the tetrad.

It is common to use the fiber-bundle formulation by which gauge theories can be constructed on the basis of any Lie group. Recent work by Hehl *et al.* [13] on the so-called Metric Affine Gravity (MAG) theory adopted as a gauge group the affine group $A(4,\,\mathbb{R}) = T(4) \rtimes GL(4,\,\mathbb{R})$ which was realized linearly. The tetrad was identified with the nonlinearly realized translational part of the affine connection on the tangent bundle. In MAG theory, the Lagrangian is quadratic in both curvature and torsion in contrast to the Einstein-Hilbert Lagrangian in general relativity which is linear in the scalar curvature. The theory has the Einstein limit on one hand and leads to the Newtonian inverse distance potential plus the linear confinement potential in the weak field approximation on the other. As we have seen above, there are many attempts to formulate gravitation as a gauge theory. Currently no theory has been uniquely accepted as the gauge theory of gravitation.

The nonlinear approach to group realizations was originally introduced by S. Coleman, J. Wess and B. Zumino [15, 16] in the context of internal symmetry groups (1969). It was later extended to the case of spacetime symmetries by Isham, Salam, and Strathdee [17, 18] considering the nonlinear action of $GL(4,\,\mathbb{R})$ mod the Lorentz subgroup. In 1974, Borisov, Ivanov and Ogievetsky [19, 20] considered the simultaneous nonlinear realization (NLR) of the affine and conformal groups. They showed that general relativity can be viewed as a consequence of spontaneous breakdown of the affine symmetry in much the same manner that chiral dynamics in quantum chromodynamics is a result of spontaneous breakdown of chiral symmetry. In their model, gravitons are considered as Goldstone bosons associated with the affine symmetry breaking. In 1978, Chang and Mansouri [21] used the NLR scheme employing $GL(4,\,\mathbb{R})$ as the principal group. In 1980, Stelle and West [22] investigated the NLR induced by the spontaneous breakdown of $SO(3,\,2)$. In 1982 Ivanov and Niederle considered nonlinear gauge theories of the Poincaré, de Sitter, conformal and special conformal groups [23, 24]. In 1983, Ivanenko and Sardanashvily [25] considered gravity to be a spontaneously broken $GL(4,\,\mathbb{R})$ gauge theory. The tetrads fields arise in their formulation as a result of the reduction of the structure group of the tangent bundle from the general linear to Lorentz group. In 1987, Lord and Goswami [26, 27] developed the NLR in the fiber bundle formalism based on the bundle structure $G\,(G/H,\,H)$ as suggested by Ne'eman and Regge [28]. In this approach the quotient space G/H is identified with physical spacetime. Most recently, in a series of papers, A. Lopez-Pinto, J. Julve, A. Tiemblo, R. Tresguerres and E. Mielke discussed nonlinear gauge theories of gravity on the basis of the Poincaré, affine and conformal groups [30, 31, 32, 33, 34, 35]. In the present paper, we consider a modified version of the theories proposed by Tresguerres and Lopez-Pinto *et al.*

The paper is organized as follows. In Section 2, mainly following Tresguerres and Tiemblo [33, 34], the generalized bundle structure of gravity is presented. In Section 3, a generalized gauge transformation law enabling the gauging of external spacetime groups is introduced. Demanding that tetrads be obtained as gauge fields requires the implementation of a NLR of the CA group. Such a NLR is carried out over the quotient space $CA(3,\,1)/SO(3,\,1)$. In Section 4, the transformations of all coset fields parameterizing this quotient space is computed. The fundamental vector field operators are computed in Section 5. In Section 6, the general form of the gauge connections of the theory along with their transformation laws are obtained. In Section 7, we present the explicit structure of the CA connections. The nonlinear translational connection coefficient (transforming as a 4-covector under the Lorentz group) is identified as a coframe field. In Section 8, the tetrad components of the coframe are used in conjunction with the Lorentz group metric to induce a spacetime metric. In Section 9, the bundle curvature of the theory together with the variations of its corresponding field strength components are determined. The Bianchi identities are obtained in Section 10. In Section 11, surface ($3D$) and bulk ($4D$) topological invariants are constructed. The bulk terms (obtained via exterior derivation of the surface terms) provide a means of "deriving" a prototype (source free) gravitational action (after appropriately distributing Lie star operators). The covariant field equations and gauge currents are obtained in Section 12. Our conclusions are presented in Section 13.

A. Ordinary Fiber Bundles, Gauge Symmetry and Connection Forms

The purpose of this section is to briefly review the standard bundle approach to gauge theories. We verify that the usual gauge potential Ω is the pullback of connection 1-form ω by local sections of the bundle. Finally, the transformation laws of the ω and Ω under the action of the structure group G are deduced.

Modern formulations of gauge field theories are expressible geometrically in the language of principal fiber bundles. A fiber bundle is a structure $\langle \mathbb{P},\, M,\, \pi;\, \mathbb{F}\rangle$ where $\mathbb{P}$ (the total bundle space) and M (the base space) are smooth

manifolds, $\mathbb{F}$ is the fiber space and the surjection π (a canonical projection) is a smooth map of $\mathbb{P}$ onto M,

$$\pi : \mathbb{P} \to M. \tag{1}$$

The inverse image π^{-1} is diffeomorphic to $\mathbb{F}$

$$\pi^{-1}(x) \equiv \mathbb{F}_x \approx \mathbb{F}, \tag{2}$$

and is called the fiber at $x \in M$. The partitioning $\bigcup_x \pi^{-1}(x) = \mathbb{P}$ is referred to as the fibration. Note that a smooth map is one whose coordinatization is C^∞ differentiable; a smooth manifold is a space that can be covered with coordinate patches in such a manner that a change from one patch to any overlapping patch is smooth, see A. S. Schwarz [36]. Fiber bundles that admit decomposition as a direct product, locally looking like $\mathbb{P} \approx M \times \mathbb{F}$, is called trivial. Given a set of open coverings $\{\mathcal{U}_i\}$ of M with $x \in \{\mathcal{U}_i\} \subset M$ satisfying $\bigcup_\alpha \mathcal{U}_\alpha = M$, the diffeomorphism map is given by

$$\chi_i : \mathcal{U}_i \times_M G \to \pi^{-1}(\mathcal{U}_i) \in \mathbb{P}, \tag{3}$$

($\times_M$ represents the fiber product of elements defined over space M) such that $\pi(\chi_i(x, g)) = x$ and $\chi_i(x, g) = \chi_i(x, (id)_G)\, g = \chi_i(x)\, g \;\forall x \in \{\mathcal{U}_i\}$ and $g \in G$. Here, $(id)_G$ represents the identity element of group G. In order to obtain the global bundle structure, the local charts χ_i must be glued together continuously. Consider two patches $\mathcal{U}_n$ and $\mathcal{U}_m$ with a non-empty intersection $\mathcal{U}_n \cap \mathcal{U}_m \neq \emptyset$. Let ρ_{nm} be the restriction of χ_n^{-1} to $\pi^{-1}(\mathcal{U}_n \cap \mathcal{U}_m)$ defined by $\rho_{nm} : \pi^{-1}(\mathcal{U}_n \cap \mathcal{U}_m) \to (\mathcal{U}_n \cap \mathcal{U}_m) \times_M G_n$. Similarly let $\rho_{mn} : \pi^{-1}(\mathcal{U}_m \cap \mathcal{U}_n) \to (\mathcal{U}_m \cap \mathcal{U}_n) \times_M G_m$ be the restriction of χ_m^{-1} to $\pi^{-1}(\mathcal{U}_n \cap \mathcal{U}_m)$. The composite diffeomorphism $\Lambda_{nm} \in G$

$$\Lambda_{mn} : (\mathcal{U}_n \cap \mathcal{U}_m) \times G_n \to (\mathcal{U}_m \cap \mathcal{U}_n) \times_M G_m, \tag{4}$$

defined as

$$\Lambda_{ij}(x) \equiv \rho_{ji} \circ \rho_{ij}^{-1} = \chi_{i,\, x} \circ \chi_{j,\, x}^{-1} : \mathbb{F} \to \mathbb{F} \tag{5}$$

constitute the transition function between bundle charts ρ_{nm} and ρ_{mn} ($\circ$ represents the group composition operation) where the diffeomorphism $\chi_{i,\, x} : \mathbb{F} \to \mathbb{F}_x$ is written as $\chi_{i,\, x}(g) := \chi_i(x, g)$ and satisfies $\chi_j(x, g) = \chi_i(x, \Lambda_{ij}(x)\, g)$. The transition functions $\{\Lambda_{ij}\}$ can be interpreted as passive gauge transformations. They satisfy the identity $\Lambda_{ii}(x)$, inverse $\Lambda_{ij}(x) = \Lambda_{ji}^{-1}(x)$ and cocycle $\Lambda_{ij}(x)\Lambda_{jk}(x) = \Lambda_{ik}(x)$ consistency conditions. For trivial bundles, the transition function reduces to

$$\Lambda_{ij}(x) = g_i^{-1} g_j, \tag{6}$$

where $g_i : \mathbb{F} \to \mathbb{F}$ is defined by $g_i := \chi_{i,\, x}^{-1} \circ \widetilde{\chi}_{i,\, x}$ provided the local trivializations $\{\chi_i\}$ and $\{\widetilde{\chi}_i\}$ give rise to the same fiber bundle.

A section is defined as a smooth map

$$s : M \to \mathbb{P}, \tag{7}$$

such that $s(x) \in \pi^{-1}(x) = \mathbb{F}_x \;\forall x \in M$ and satisfies

$$\pi \circ s = (id)_M\,, \tag{8}$$

where $(id)_M$ is the identity element of M. It assigns to each point $x \in M$ a point in the fiber over x. Trivial bundles admit global sections.

A bundle is a principal fiber bundle $\langle \mathbb{P}, \mathbb{P}/G, G, \pi \rangle$ provided the Lie group G acts freely (i.e. if $pg = p$ then $g = (id)_G$) on $\mathbb{P}$ to the right $R_g p = pg$, $p \in \mathbb{P}$, preserves fibers on $\mathbb{P}$ ($R_g : \mathbb{P} \to \mathbb{P}$), and is transitive on fibers. Furthermore, there must exist local trivializations compatible with the G action. Hence, $\pi^{-1}(\mathcal{U}_i)$ is homeomorphic to $\mathcal{U}_i \times_M G$ and the fibers of $\mathbb{P}$ are diffeomorphic to G. The trivialization or inverse diffeomorphism map is given by

$$\chi_i^{-1} : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times_M G \tag{9}$$

such that $\chi^{-1}(p) = (\pi(p), \varphi(p)) \in \mathcal{U}_i \times_M G$, $p \in \pi^{-1}(\mathcal{U}_i) \subset \mathbb{P}$, where we see from the above definition that φ is a local mapping of $\pi^{-1}(\mathcal{U}_i)$ into G satisfying $\varphi(L_g p) = \varphi(p) g$ for any $p \in \pi^{-1}(\mathcal{U})$ and any $g \in G$. Observe that the elements of $\mathbb{P}$ which are projected onto the same $x \in \{\mathcal{U}_i\}$ are transformed into one another by the elements of G. In other words, the fibers of $\mathbb{P}$ are the orbits of G and at the same time, the set of elements which are projected onto the same

$x \in \mathcal{U} \subset M$. This observation motivates calling the action of the group vertical and the base manifold horizontal. The diffeomorphism map χ_i is called the local gauge since χ_i^{-1} maps $\pi^{-1}(\mathcal{U}_i)$ onto the direct (Cartesian) product $\mathcal{U}_i \times_M G$. The action L_g of the structure group G on $\mathbb{P}$ defines an isomorphism of the Lie algebra $\mathfrak{g}$ of G onto the Lie algebra of vertical vector fields on $\mathbb{P}$ tangent to the fiber at each $p \in \mathbb{P}$ called fundamental vector fields

$$\lambda_g : T_p(\mathbb{P}) \to T_{gp}(\mathbb{P}) = T_{\pi(p)}(\mathbb{P}), \tag{10}$$

where $T_p(\mathbb{P})$ is the space of tangents at p, i.e. $T_p(\mathbb{P}) \in T(\mathbb{P})$. The map λ is a linear isomorphism for every $p \in \mathbb{P}$ and is invariant with respect to the action of G, that is, $\lambda_g : (\lambda_{g*} T_p(\mathbb{P})) \to T_{gp}(\mathbb{P})$, where λ_{g*} is the differential push forward map induced by λ_g defined by $\lambda_{g*} : T_p(\mathbb{P}) \to T_{gp}(\mathbb{P})$.

Since the principal bundle $\mathbb{P}(M, G)$ is a differentiable manifold, we can define tangent $T(\mathbb{P})$ and cotangent $T^*(\mathbb{P})$ bundles. The tangent space $T_p(\mathbb{P})$ defined at each point $p \in \mathbb{P}$ may be decomposed into a vertical $V_p(\mathbb{P})$ and horizontal $H_p(\mathbb{P})$ subspace as $T_p(\mathbb{P}) := V_p(\mathbb{P}) \oplus H_p(\mathbb{P})$ (where $\oplus$ represents the direct sum). The space $V_p(\mathbb{P})$ is a subspace of $T_p(\mathbb{P})$ consisting of all tangent vectors to the fiber passing through $p \in \mathbb{P}$, and $H_p(\mathbb{P})$ is the subspace complementary to $V_p(\mathbb{P})$ at p. The vertical subspace $V_p(\mathbb{P}) := \{X \in T(\mathbb{P}) \,|\pi(X) \in \mathcal{U}_i \subset M\}$ is uniquely determined by the structure of $\mathbb{P}$, whereas the horizontal subspace $H_p(\mathbb{P})$ cannot be uniquely specified. Thus we require the following condition: when p transforms as $p \to p' = pg$, $H_p(\mathbb{P})$ transforms as [37],

$$R_{g*} H_p(\mathbb{P}) \to H_{p'}(\mathbb{P}) = R_g H_p(\mathbb{P}) = H_{pg}(\mathbb{P}). \tag{11}$$

Let the local coordinates of $\mathbb{P}(M, G)$ be $p = (x, g)$ where $x \in M$ and $g \in G$. Let $\mathbf{G}_A$ denote the generators of the Lie algebra $\mathfrak{g}$ corresponding to group G satisfying the commutators $[\mathbf{G}_A, \mathbf{G}_B] = f_{AB}{}^C \mathbf{G}_C$, where $f_{AB}{}^C$ are the structure constants of G. Let Ω be a connection form defined by $\Omega^A := \Omega_i^A dx^i \in \mathfrak{g}$. Let ω be a connection 1-form defined by

$$\omega := \widetilde{g}^{-1} \pi^*_{\mathbb{P}M} \Omega \widetilde{g} + \widetilde{g}^{-1} d\widetilde{g} \tag{12}$$

($*$ represents the differential pullback map) belonging to $\mathfrak{g} \otimes T_p^*(\mathbb{P})$ where $T_p^*(\mathbb{P})$ is the space dual to $T_p(\mathbb{P})$. The differential pullback map applied to a test function φ and p-forms α and β satisfy $f^*\varphi = \varphi \circ f$, $(g \circ f)^* = f^* g^*$ and $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$. If G is represented by a d-dimensional $d \times d$ matrix, then $\mathbf{G}_A = [\mathbf{G}_{\alpha\beta}]$, $\widetilde{g} = [\widetilde{g}^{\alpha\beta}]$, where α, $\beta = 1, 2, 3, \ldots d$. Thus, ω assumes the form

$$\omega_\alpha^{\ \beta} = \left(\widetilde{g}^{-1}\right)_{\alpha\gamma} d\widetilde{g}^{\gamma\beta} + \left(\widetilde{g}^{-1}\right)_{\rho\gamma} \pi^*_{\mathbb{P}M} \Omega^\rho_{\ \sigma i} \mathbf{G}_\alpha^{\ \gamma} \widetilde{g}^{\sigma\beta} \otimes dx^i. \tag{13}$$

If M is n-dimensional, the tangent space $T_p(\mathbb{P})$ is $(n+d)$-dimensional. Since the vertical subspace $V_p(\mathbb{P})$ is tangential to the fiber G, it is d-dimensional. Accordingly, $H_p(\mathbb{P})$ is n-dimensional. The basis of $V_p(\mathbb{P})$ can be taken to be $\partial_{\alpha\beta} := \frac{\partial}{\partial g^{\alpha\beta}}$. Now, let the basis of $H_p(\mathbb{P})$ be denoted by

$$E_i := \partial_i + \Gamma_i^{\alpha\beta} \partial_{\alpha\beta},\ i = 1, 2, 3, ..n \text{ and } \alpha,\ \beta = 1, 2, 3, ..d \tag{14}$$

where $\partial_i = \frac{\partial}{\partial x^i}$. The connection 1-form ω projects $T_p(\mathbb{P})$ onto $V_p(\mathbb{P})$. In order for $X \in T_p(\mathbb{P})$ to belong to $H_p(\mathbb{P})$, that is for $X \in H_p(\mathbb{P})$, $\omega_p(X) = \langle \omega(p) \,| X \rangle = 0$. In other words,

$$H_p(\mathbb{P}) := \{X \in T_p(\mathbb{P}) \,| \omega_p(X) = 0\}, \tag{15}$$

from which $\Omega_i^{\alpha\beta}$ can be determined. The inner product appearing in $\omega_p(X) = \langle \omega(p) \,| X \rangle = 0$ is a map $\langle \cdot | \cdot \rangle : T_p^*(\mathbb{P}) \times T_p(\mathbb{P}) \to \mathbb{R}$ defined by $\langle W | V \rangle = W_\mu V^\nu \left\langle dx^\mu | \frac{\partial}{\partial x^\nu} \right\rangle = W_\mu V^\nu \delta^\mu_\nu$, where the 1-form W and vector V are given by $W = W_\mu dx^\mu$ and $V = V^\mu \frac{\partial}{\partial x^\nu}$. Observe also that, $\left\langle dg^{\alpha\beta} | \partial_{\rho\sigma} \right\rangle = \delta^\alpha_\rho \delta^\beta_\sigma$.

We parameterize an arbitrary group element $\widetilde{g}_\lambda$ as $\widetilde{g}(\lambda) = e^{\lambda^A \mathbf{G}_A} = e^{\lambda \cdot \mathbf{G}}$, $A = 1, ..dim(\mathfrak{g})$. The right action $R_{\widetilde{g}(\lambda)} = R_{\exp(\lambda \cdot G)}$ on $p \in \mathbb{P}$, i.e. $R_{\exp(\lambda \cdot \mathbf{G})} p = p \exp(\lambda \cdot \mathbf{G})$, defines a curve through p in $\mathbb{P}$. Define a vector $G^\# \in T_p(\mathbb{P})$ by [37]

$$G^\# f(p) := \frac{d}{dt} f(p \exp(\lambda \cdot \mathbf{G})) \,|_{\lambda=0} \tag{16}$$

where $f : \mathbb{P} \to \mathbb{R}$ is an arbitrary smooth function. Since the vector $G^\#$ is tangent to $\mathbb{P}$ at p, $G^\# \in V_p(\mathbb{P})$, the components of the vector $G^\#$ are the fundamental vector fields at p which constitute $V(\mathbb{P})$. The components of $G^\#$ may also be viewed as a basis element of the Lie algebra $\mathfrak{g}$. Given $G^\# \in V_p(\mathbb{P})$, $\mathbf{G} \in \mathfrak{g}$,

$$\begin{aligned} \omega_p\left(G^\#\right) &= \left\langle \omega(p) \,| G^\# \right\rangle = \widetilde{g}^{-1} d\widetilde{g}\left(G^\#\right) + \widetilde{g}^{-1} \pi^*_{\mathbb{P}M} \Omega \widetilde{g}\left(G^\#\right) \\ &= \widetilde{g}_p^{-1} \widetilde{g}_p \frac{d}{d\lambda} \left(\exp(\lambda \cdot \mathbf{G})\right) |_{\lambda=0}, \end{aligned} \tag{17}$$

where use was made of $\pi_{\mathbb{P}M*}G^{\#}=0$. Hence, $\omega_p\left(G^{\#}\right)=\mathbf{G}$. An arbitrary vector $X\in H_p\left(\mathbb{P}\right)$ may be expanded in a basis spanning $H_p\left(\mathbb{P}\right)$ as $X:=\beta^iE_i$. By direct computation, one can show

$$\left\langle \omega_{\alpha}^{\ \beta}|X\right\rangle=\left(\widetilde{g}^{-1}\right)_{\alpha\gamma}\beta^i\Gamma_i^{\gamma\beta}+\left(\widetilde{g}^{-1}\right)_{\alpha\gamma}\pi_{\mathbb{P}M}^{*}\Omega^{\rho}{}_{\sigma i}\beta^i\mathbf{G}_{\rho}^{\gamma}\widetilde{g}^{\sigma\beta}=0,\ \forall\beta^i \tag{18}$$

Equation (18) yields

$$\left(\widetilde{g}^{-1}\right)_{\alpha\gamma}\Gamma_i^{\gamma\beta}+\left(\widetilde{g}^{-1}\right)_{\alpha\gamma}\pi_{\mathbb{P}M}^{*}\Omega^{\rho}{}_{\sigma i}\mathbf{G}_{\rho}^{\gamma}\widetilde{g}^{\sigma\beta}=0, \tag{19}$$

from which we obtain

$$\Gamma_i^{\gamma\beta}=-\pi_{\mathbb{P}M}^{*}\Omega^{\rho}{}_{\sigma i}\mathbf{G}_{\rho}^{\gamma}\widetilde{g}^{\sigma\beta}. \tag{20}$$

In this manner, the horizontal component is completely determined. An arbitrary tangent vector $\mathfrak{X}\in T_p\left(\mathbb{P}\right)$ defined at $p\in\mathbb{P}$ takes the form

$$\mathfrak{X}=A^{\alpha\beta}\partial_{\alpha\beta}+B^i\left(\partial_i-\pi_{\mathbb{P}M}^{*}\Omega^{\rho}{}_{\sigma i}\mathbf{G}_{\rho}^{\alpha}\widetilde{g}^{\sigma\beta}\partial_{\alpha\beta}\right), \tag{21}$$

where $A^{\alpha\beta}$ and B^i are constants. The vector field $\mathfrak{X}$ is comprised of horizontal $\mathfrak{X}^H:=B^i\left(\partial_i-\pi_{\mathbb{P}M}^{*}\Omega^{\rho}{}_{\sigma i}\mathbf{G}_{\rho}^{\alpha}\widetilde{g}^{\sigma\beta}\partial_{\alpha\beta}\right)\in H\left(\mathbb{P}\right)$ and vertical $\mathfrak{X}^V:=A^{\alpha\beta}\partial_{\alpha\beta}\in V\left(\mathbb{P}\right)$ components.

Let $\mathfrak{X}\in T_p\left(\mathbb{P}\right)$ and $g\in\mathbf{G}$, then

$$R_g^{*}\omega\left(\mathfrak{X}\right)=\omega\left(R_{g*}\mathfrak{X}\right)=\widetilde{g}_{pg}^{-1}\Omega\left(R_{g*}\mathfrak{X}\right)\widetilde{g}_{pg}+\widetilde{g}_{pg}^{-1}d\widetilde{g}_{pg}\left(R_{g*}\mathfrak{X}\right), \tag{22}$$

Observing that $\widetilde{g}_{pg}=\widetilde{g}_pg$ and $\widetilde{g}_{gp}^{-1}=g^{-1}\widetilde{g}_p^{-1}$ the first term on the RHS of (22) reduces to $\widetilde{g}_{pg}^{-1}\Omega\left(R_{g*}\mathfrak{X}\right)\widetilde{g}_{pg}=g^{-1}\widetilde{g}_p^{-1}\Omega\left(R_{g*}\mathfrak{X}\right)\widetilde{g}_pg$ while the second term gives $\widetilde{g}_{pg}^{-1}d\widetilde{g}_{pg}\left(R_{g*}\mathfrak{X}\right)=g^{-1}\widetilde{g}_p^{-1}d\left(R_{g*}\mathfrak{X}\right)\widetilde{g}_pg$. We therefore conclude

$$R_g^{*}\omega_{\lambda}=ad_{g^{-1}}\omega_{\lambda}, \tag{23}$$

where the adjoint map ad is defined by

$$ad_gY:=L_{g*}\circ R_{g^{-1}*}\circ Y=gYg^{-1},\ \ ad_{g^{-1}}Y:=g^{-1}Yg. \tag{24}$$

The potential Ω^A can be obtained from ω as $\Omega^A=s^{*}\omega$. To demonstrate this, let $Y\in T_p\left(M\right)$ and $\widetilde{g}$ be specified by the inverse diffeomorphism or trivialization map (9) with $\chi_{\lambda}^{-1}\left(p\right)=\left(x,\,\widetilde{g}_{\lambda}\right)$ for $p\left(x\right)=s_{\lambda}\left(x\right)\cdot\widetilde{g}_{\lambda}$. We find $s_i^{*}\omega\left(Y\right)=\widetilde{g}^{-1}\Omega\left(\pi_{*}s_{i*}Y\right)\widetilde{g}+\widetilde{g}^{-1}d\widetilde{g}\left(s_{i*}Y\right)$, where we [37] have used $s_{i*}Y\in T_{s_i}\left(\mathbb{P}\right)$, $\pi_{*}s_{i*}=\left(id\right)_{T_p(M)}$ and $\widetilde{g}=\left(id\right)_G$ at s_i implying $\widetilde{g}^{-1}d\widetilde{g}\left(s_{i*}Y\right)=0$. Hence,

$$s_i^{*}\omega\left(Y\right)=\Omega\left(Y\right). \tag{25}$$

To determine the gauge transformation of the connection 1-form ω we use the fact that $R_{\widetilde{g}*}X=X\widetilde{g}$ for $X\in T_p\left(M\right)$ and the transition functions $\widetilde{g}_{nm}\in G$ defined between neighboring bundle charts (6). By direct computation we get

$$\begin{aligned} c_{j*}X &= \frac{d}{dt}c_j\left(\lambda\left(t\right)\right)|_{t=0}=\frac{d}{dt}\left[c_i\left(\lambda\left(t\right)\right)\cdot\widetilde{g}_{ij}\right]|_{t=0} \\ &= R_{\widetilde{g}_{ij}*}c_i^{*}\left(X\right)+\left(\widetilde{g}_{ji}^{-1}\left(x\right)d\widetilde{g}_{ij}\left(X\right)\right)^{\#}. \end{aligned} \tag{26}$$

where $\lambda\left(t\right)$ is a curve in M with boundary values $\lambda\left(0\right)=m$ and $\frac{d}{dt}\lambda\left(t\right)|_{t=0}=X$. Thus, we obtain the useful result

$$c_{*}X=R_{\widetilde{g}*}\left(c_{*}X\right)+\left(\widetilde{g}^{-1}d\widetilde{g}\left(X\right)\right)^{\#}. \tag{27}$$

Applying ω to (27) we get

$$\omega\left(c_{*}X\right)=c^{*}\omega\left(X\right)=ad_{\widetilde{g}^{-1}}c^{*}\omega\left(X\right)+\widetilde{g}^{-1}d\widetilde{g}\left(X\right),\ \forall X. \tag{28}$$

Hence, the gauge transformation of the local gauge potential Ω reads,

$$\Omega\rightarrow\Omega'=ad_{\widetilde{g}^{-1}}\left(d+\Omega\right)=\widetilde{g}^{-1}\left(d+\Omega\right)\widetilde{g}. \tag{29}$$

Since $\Omega=c^{*}\omega$ we obtain from (29) the gauge transformation law of ω

$$\omega\rightarrow\omega'=\widetilde{g}^{-1}\left(d+\omega\right)\widetilde{g}. \tag{30}$$

II. GENERALIZED BUNDLE STRUCTURE OF GRAVITATION

Let us recall the definition of gauge transformations in the context of ordinary fiber bundles. Given a principal fiber bundle $\mathbb{P}(M,\,G;\,\pi)$ with base space M and standard G-diffeomorphic fiber, gauge transformations are characterized by bundle isomorphisms [39] $\lambda:\mathbb{P}\to\mathbb{P}$ exhausting all diffeomorphisms λ_M on M. This mapping is called an automorphism of $\mathbb{P}$ provided it is equivariant with respect to the action of G. This amounts to restricting the action λ of G along local fibers leaving the base space unaffected. Indeed, with regard to gauge theories of internal symmetry groups, a gauge transformation is a fiber preserving bundle automorphism, i.e. diffeomorphisms λ with $\lambda_M=(id)_M$. The automorphisms λ form a group called the automorphism group $Aut_{\mathbb{P}}$ of $\mathbb{P}$. The gauge transformations form a subgroup of $Aut_{\mathbb{P}}$ called the gauge group $G\,(Aut_{\mathbb{P}})$ (or G in short) of $\mathbb{P}$.

The map λ is required to satisfy two conditions, namely its commutability with the right action of G [the equivariance condition $\lambda\,(R_g(p))=\lambda\,(pg)=\lambda\,(p)\,g$]

$$\lambda\circ R_g(p)=R_g(p)\circ\lambda,\;\; p\in\mathbb{P},\, g\in G \tag{31}$$

according to which fibers are mapped into fibers, and the verticality condition

$$\pi\circ\lambda\,(u)=\pi\,(u)\,, \tag{32}$$

where u and $\lambda\,(u)$ belong to the same fiber. The last condition ensures that no diffeomorphisms $\lambda_M:M\to M$ given by

$$\lambda_M\circ\pi\,(u)=\pi\circ\lambda\,(u)\,, \tag{33}$$

be allowed on the base space M. In a gauge description of gravitation, one is interested in gauging external transformation groups. That is to say the group action on spacetime coordinates cannot be neglected. The spaces of internal fiber and external base must be interlocked in the sense that transformations in one space must induce corresponding transformations in the other. The usual definition of a gauge transformation, i.e. as a displacement along local fibers not affecting the base space, must be generalized to reflect this interlocking. One possible way of framing this interlocking is to employ a nonlinear realization of the gauge group G, provided a closed subgroup $H\subset G$ exist. The interlocking requirement is then transformed into the interplay between groups G and one of its closed subgroups H.

Denote by G a Lie group with elements $\{g\}$. Let H be a closed subgroup of G specified by [37, 67]

$$H:=\{h\in G|\Pi\,(R_h g)=\pi\,(g)\,,\,\forall g\in G\}\,, \tag{34}$$

with elements $\{h\}$ and known linear representations $\rho\,(h)$. Here Π is the first of the two projection maps in (37), and R_h is the right group action. Let M be a differentiable manifold with points $\{x\}$ to which G and H may be referred, i.e. $g=g(x)$ and $h=h(x)$. Being that G and H are Lie groups, they are also manifolds. The right action of H on G induce a complete partition of G into mutually disjoint orbits gH. Since $g=g(x)$, all elements of $gH=\{gh_1,\,gh_2,\,gh_3,\cdots,\,gh_n\}$ are defined over the same x. Thus, each orbit gH constitute an equivalence class of point x, with equivalence relation $g\equiv g'$ where $g'=R_h g=gh$. By projecting each equivalence class onto a single element of the quotient space $\mathcal{M}:=G/H$, the group G becomes organized as a fiber bundle in the sense that $G=\bigcup_i\{g_i H\}$. In this manner the manifold G is viewed as a fiber bundle $G\,(\mathcal{M},\,H;\,\Pi)$ with H-diffeomorphic fibers $\Pi^{-1}\,(\xi):G\to\mathcal{M}=gH$ and base space $\mathcal{M}$. A composite principal fiber bundle $\mathbb{P}(M,\,G;\,\pi)$ is one whose G-diffeomorphic fibers possess the fibered structure $G\,(\mathcal{M},\,H;\,\Pi)\simeq\mathcal{M}\times\,H$ described above. The bundle $\mathbb{P}$ is then locally isomorphic to $M\times G\,(\mathcal{M},\,H)$. Moreover, since an element $g\in G$ is locally homeomorphic to $\mathcal{M}\times H$ the elements of $\mathbb{P}$ are - by transitivity - also locally homeomorphic to $M\times\mathcal{M}\times H\simeq\Sigma\times H$ where (locally) $\Sigma\simeq M\times\mathcal{M}$. Thus, an alternative view [33] of $\mathbb{P}(M,\,G;\,\pi)$ is provided by the $\mathbb{P}$-associated H-bundle $\mathbb{P}(\Sigma,\,H;\,\widetilde{\pi})$. The total space $\mathbb{P}$ may be regarded as $G\,(\mathcal{M},\,H;\,\Pi)$-bundles over base space M or equivalently as H-fibers attached to manifold $\Sigma\simeq M\times\mathcal{M}$.

The nonlinear realization (NLR) technique [15, 16] provides a way to determine the transformation properties of fields defined on the quotient space G/H. The NLR of Diff(4, $\mathbb{R}$) becomes tractable due to a theorem given by V. I. Ogievetsky. According to the Ogievetsky theorem [19], the algebra of the infinite dimensional group Diff(4, $\mathbb{R}$) can be taken as the closure of the finite dimensional algebras of $SO(4,\,2)$ and $A(4,\,\mathbb{R})$. Remind that the Lorentz group generates transformations that preserve the quadratic form on Minkowski spacetime built from the metric tensor, while the special conformal group generates infinitesimal angle-preserving transformations on Minkowski spacetime. The affine group is a generalization of the Poincaré group where the Lorentz group is replaced by the group of general linear transformations. As such, the affine group generates translations, Lorentz transformations, volume preserving shear and volume changing dilation transformations. As a consequence, the NLR of Diff$(4,\,\mathbb{R})\,/SO(3,\,1)$ can be constructed by taking a simultaneous realization of the conformal group $SO(4,\,2)$ and the affine group $A(4,$

$\mathbb{R}) := \mathbb{R}^4 \rtimes GL(4,\, \mathbb{R})$ on the coset spaces $A(4,\, \mathbb{R})/SO(3,\, 1)$ and $SO(4,\, 2)/SO(3,\, 1)$. One possible interpretation of this theorem is that the conform-affine group (defined below) may be the largest subgroup of Diff(4, $\mathbb{R}$) whose transformations may be put into the form of a generalized coordinate transformation. We remark that a NLR can be made linear by embedding the representation in a sufficiently higher dimensional space. Alternatively, a linear group realization becomes nonlinear when subject to constraints. One type of relevant constraints may be those responsible for symmetry reduction from Diff(4, $\mathbb{R}$) to $SO(3,\, 1)$ for instance.

We take the group $CA(3,\, 1)$ as the basic symmetry group G. The CA group consists of the groups $SO(4,\, 2)$ and $A(4,\, \mathbb{R})$. In particular, CA is proportional to the union $SO(4,\, 2) \cup A(4,\, \mathbb{R})$. We know however (see section *Conform-Affine Lie Algebra*) that the affine and special conformal groups have several group generators in common. These common generators reside in the intersection $SO(4,\, 2) \cap A(4,\, \mathbb{R})$ of the two groups, within which there are *two copies* of $\Pi := D \times P(3,\, 1)$, where D is the group of scale transformations (dilations) and $P(3,\, 1) := T\,(3,\, 1) \rtimes SO(3,\, 1)$ is the Poincaré group. We define the CA group as the union of the affine and conformal groups minus *one copy* of the overlap Π, i.e. $CA(3,\, 1) := SO(4,\, 2) \cup A(4,\, \mathbb{R}) - \Pi$. Being defined in this way we recognize that $CA(3,\, 1)$ is a 24 parameter Lie group representing the action of Lorentz transformations (6), translations (4), special conformal transformations (4), spacetime shears (9) and scale transformations (1). In this paper, we obtain the NLR of $CA(3,\, 1)$ modulo $SO(3,\, 1)$.

A. Conform-Affine Lie Algebra

In order to implement the NLR procedure, we choose to partition Diff(4, $\mathbb{R}$) with respect to the Lorentz group. By Ogievetsky's theorem [19], we identify representations of Diff(4, $\mathbb{R})/SO(3,\, 1)$ with those of $CA(3,\, 1)/SO(3,\, 1)$. The 20 generators of affine transformations can be decomposed into the 4 translational $\mathbf{P}^{\mathrm{Aff}}_{\mu}$ and 16 $GL(4,\, \mathbb{R})$ transformations $\mathbf{\Lambda}_{\alpha}^{\ \beta}$. The 16 generators $\mathbf{\Lambda}_{\alpha}^{\ \beta}$ may be further decomposed into the 6 Lorentz generators $\mathbf{L}_{\alpha}^{\ \beta}$ plus the remaining 10 generators of symmetric linear transformation $\mathbf{S}_{\alpha}^{\ \beta}$, that is, $\mathbf{\Lambda}^{\alpha}_{\ \beta} = \mathbf{L}^{\alpha}_{\ \beta} + \mathbf{S}^{\alpha}_{\ \beta}$. The 10 parameter symmetric linear generators $\mathbf{S}_{\alpha}^{\ \beta}$ can be factored into the 9 parameter shear (the traceless part of $\mathbf{S}_{\alpha}^{\ \beta}$) generator defined by ${}^{\dagger}\mathbf{S}_{\alpha}^{\ \beta} = \mathbf{S}_{\alpha}^{\ \beta} - \frac{1}{4}\delta_{\alpha}^{\ \beta}\mathbf{D}$, and the 1 parameter dilaton generator $\mathbf{D} = tr\left(\mathbf{S}_{\alpha}^{\ \beta}\right)$. Shear transformations generated by ${}^{\dagger}\mathbf{S}_{\alpha}^{\ \beta}$ describe shape changing, volume preserving deformations, while the dilaton generator gives rise to volume changing transformations. The four diagonal elements of $\mathbf{S}_{\alpha}^{\ \beta}$ correspond to the generators of projective transformations. The 15 generators of conformal transformations are defined in terms of the set $\{J_{AB}\}$ where $A = 0,\, 1,\, 2,..5$. The elements J_{AB} can be decomposed into translations $\mathbf{P}^{\mathrm{Conf}}_{\mu} := J_{5\mu} + J_{6\mu}$, special conformal generators $\mathbf{\Delta}_{\mu} := J_{5\mu} - J_{6\mu}$, dilatons $\mathbf{D} := J_{56}$ and the Lorentz generators $\mathbf{L}_{\alpha\beta} := J_{\alpha\beta}$. The Lie algebra of $CA(3,\, 1)$ is characterized by the commutation relations

$$\begin{gathered}
[\mathbf{\Lambda}_{\alpha\beta},\, \mathbf{D}] = [\mathbf{\Delta}_{\alpha},\, \mathbf{\Delta}_{\beta}] = 0,\ [\mathbf{P}_{\alpha},\, \mathbf{P}_{\beta}] = [\mathbf{D},\, \mathbf{D}] = 0,\\
[\mathbf{L}_{\alpha\beta},\, \mathbf{P}_{\mu}] = io_{\mu[\alpha}\mathbf{P}_{\beta]},\ [\mathbf{L}_{\alpha\beta},\, \mathbf{\Delta}_{\gamma}] = io_{[\alpha|\gamma}\mathbf{\Delta}_{|\beta]},\\
\left[\mathbf{\Lambda}^{\alpha}_{\ \beta},\, \mathbf{P}_{\mu}\right] = i\delta^{\alpha}_{\mu}\mathbf{P}_{\beta},\ \left[\mathbf{\Lambda}^{\alpha}_{\ \beta},\, \mathbf{\Delta}_{\mu}\right] = i\delta^{\alpha}_{\mu}\mathbf{\Delta}_{\beta},\\
[\mathbf{S}_{\alpha\beta},\, \mathbf{P}_{\mu}] = io_{\mu(\alpha}\mathbf{P}_{\beta)},\ [\mathbf{P}_{\alpha},\, \mathbf{D}] = -i\mathbf{P}_{\alpha},\\
[\mathbf{L}_{\alpha\beta},\, \mathbf{L}_{\mu\nu}] = -i\left(o_{\alpha[\mu}\mathbf{L}_{\nu]\beta} - o_{\beta[\mu}\mathbf{L}_{\nu]\alpha}\right),\\
[\mathbf{S}_{\alpha\beta},\, \mathbf{S}_{\mu\nu}] = i\left(o_{\alpha(\mu}\mathbf{L}_{\nu)\beta} - o_{\beta(\mu}\mathbf{L}_{\nu)\alpha}\right),\\
[\mathbf{L}_{\alpha\beta},\, \mathbf{S}_{\mu\nu}] = i\left(o_{\alpha(\mu}\mathbf{S}_{\nu)\beta} - o_{\beta(\mu}\mathbf{S}_{\nu)\alpha}\right),\\
[\mathbf{\Delta}_{\alpha},\, \mathbf{D}] = i\mathbf{\Delta}_{\alpha},\ [\mathbf{S}_{\mu\nu},\, \mathbf{\Delta}_{\alpha}] = io_{\alpha(\mu}\mathbf{\Delta}_{\nu)},\\
\left[\mathbf{\Lambda}^{\alpha}_{\ \beta},\, \mathbf{\Lambda}^{\mu}_{\ \nu}\right] = i\left(\delta^{\alpha}_{\nu}\mathbf{\Lambda}^{\mu}_{\ \beta} - \delta^{\mu}_{\beta}\mathbf{\Lambda}^{\alpha}_{\ \nu}\right),\\
[\mathbf{P}_{\alpha},\, \mathbf{\Delta}_{\beta}] = 2i\left(o_{\alpha\beta}\mathbf{D} - \mathbf{L}_{\alpha\beta}\right),
\end{gathered} \tag{35}$$

where $o_{\alpha\beta} = diag\,(-1,\, 1,\, 1,\, 1)$ is Lorentz group metric.

III. GROUP ACTIONS AND BUNDLE MORPHISMS

In this section we introduce the main ingredients required to specify the structure of the fiber bundle we employ, namely the canonical projection, sections etc. Our main guide in this section is Tresguerres [33]. We follow his prescription for constructing the composite fiber bundle, but implement the program for the CA group.

The composite bundle $\mathbb{P}(\Sigma,\, H;\, \widetilde{\pi})$ is comprised of H-fibers, base space $\Sigma\,(M,\, \mathcal{M})$ and a composite map

$$\widetilde{\pi} \stackrel{\mathrm{def}}{=} \widetilde{\pi}_{\Sigma M} \circ \Pi_{\mathbb{P}\Sigma} : \mathbb{P} \to \Sigma \to M, \tag{36}$$

with component projections

$$\Pi_{\mathbb{P}\Sigma}:\mathbb{P}\to\Sigma,\;\;\widetilde{\pi}_{\Sigma M}:\Sigma\to M. \tag{37}$$

The projection $\Pi_{\mathbb{P}\Sigma}$ maps the point $(p\in\mathbb{P},\,R_h p\in\mathbb{P})$ into point $(x,\,\xi)\in\Sigma$. There is a correspondence between sections $s_{M\Sigma}:M\to\Sigma$ and the projection $\Pi_{\mathbb{P}\Sigma}:\mathbb{P}\to\Sigma$ in the sense that both maps project their functional argument onto elements of Σ. This is formalized by the relation, $\Pi_{\mathbb{P}\Sigma}(p)=s_{M\Sigma}\circ\pi_{\mathbb{P}M}(p)$. Hence, the total projection is given by

$$\widetilde{\pi}:=\pi_{\mathbb{P}M}=\widetilde{\pi}_{\Sigma M}\circ\Pi_{\mathbb{P}\Sigma}. \tag{38}$$

Associated with the projections $\widetilde{\pi}_{\Sigma M}$ and $\Pi_{\mathbb{P}\Sigma}$ are the corresponding local sections

$$s_{M\Sigma}:\mathcal{U}\to\widetilde{\pi}^{-1}_{\Sigma M}(\mathcal{U})\subset\Sigma,\;s_{\Sigma\mathbb{P}}:\mathcal{V}\to\Pi^{-1}_{\mathbb{P}\Sigma}(\mathcal{V})\subset\mathbb{P}, \tag{39}$$

with neighborhoods $\mathcal{U}\subset M$ and $\mathcal{V}\subset\Sigma$ satisfying

$$\widetilde{\pi}_{\Sigma M}\circ s_{M\Sigma}=(id)_M\,,\;\Pi_{\mathbb{P}\Sigma}\circ s_{\Sigma\mathbb{P}}=(id)_\Sigma\,. \tag{40}$$

The bundle injection $\widetilde{\pi}^{-1}(\mathcal{U})$ is the inverse image of $\widetilde{\pi}(\mathcal{U})$ and is called the fiber over $\mathcal{U}$. The equivalence class $R_h p=pH\in\widetilde{\pi}^{-1}_{\Sigma M}(\mathcal{U})$ of left cosets is the fiber of $\mathbb{P}(\Sigma,\,H)$ while each orbit pH through $p\in\mathbb{P}$ projects into a single element $Q\in\Sigma$. In analogy to the total bundle projection (37), a total section of $\mathbb{P}$ is given by the total section composition

$$s_{M\mathbb{P}}=s_{\Sigma\mathbb{P}}\circ s_{M\Sigma}. \tag{41}$$

Let elements of G/H be labeled by the parameter ξ. Functions on G/H are represented by continuous coset functions $c(\xi)$ parameterized by ξ. These elements are referred to as cosets to the right of H with respect to $g\in G$. Indeed, the orbits of the right action of H on G are the left cosets $R_h g=gH$. For a given section $s_{M\mathbb{P}}(x\in M)\in\pi^{-1}_{\mathbb{P}M}$ with local coordinates $(x,\,g)$ one can perform decompositions of the partial fibers $s_{M\Sigma}$ and $s_{\Sigma\mathbb{P}}$ as:

$$s_{M\Sigma}(x)=\widetilde{c}_{M\Sigma}(x)\cdot c=R_{c'}\circ\widetilde{c}_{M\Sigma}(x)\,;\;c=c(\xi)\,, \tag{42}$$

$$s_{\Sigma\mathbb{P}}(x,\,\xi)=\widetilde{c}_{\Sigma\mathbb{P}}(x,\,\xi)\cdot a'=R_{a'}\circ\widetilde{c}_{\Sigma\mathbb{P}}(x,\,\xi)\,;\;a'\in H, \tag{43}$$

with the null sections $\{\widetilde{c}_{M\Sigma}(x)\}$ and $\{\widetilde{c}_{\Sigma\mathbb{P}}(x,\,\xi)\}$ having coordinates $(x,\;(id)_{\mathcal{M}})$ and $(x,\,\xi,\;(id)_H)$ respectively. A null or zero section is a map that sends every point $x\in M$ to the origin of the fiber $\pi^{-1}(x)$ over x, i.e. $\chi_i^{-1}(\widetilde{c}(x))=(x,\,0)$ in any trivialization. The trivialization map χ_i^{-1} is defined in (9) The identity map appearing in the above trivializations are defined as $(id)_{\mathcal{M}}:\mathcal{M}\to\mathcal{M}$ and $(id)_H:H\to H$. We assume the total null bundle section be given by the composition law

$$\widetilde{c}_{M\mathbb{P}}=\widetilde{c}_{\Sigma\mathbb{P}}\circ\widetilde{c}_{M\Sigma}. \tag{44}$$

The images of two sections $s_{\Sigma\mathbb{P}}$ and $s_{M\Sigma}$ over $x\in M$ must coincide, implying $s_{\Sigma\mathbb{P}}(x,\,\xi)=s_{M\Sigma}(x)$. Using (41) with (42), (43) and (44), we arrive at the total bundle section decomposition

$$s_{M\mathbb{P}}(x)=\widetilde{c}_{M\mathbb{P}}(x)\cdot g=R_g\circ\widetilde{c}_{M\mathbb{P}}(x) \tag{45}$$

provided $g=c\cdot a$ and

$$\widetilde{c}_{\Sigma\mathbb{P}}=R_{c^{-1}}\circ\widetilde{c}_{\Sigma\mathbb{P}}(x,\,\xi)\circ R_c. \tag{46}$$

The pullback of $\widetilde{c}_{\Sigma\mathbb{P}}$, defined [33] as

$$\widetilde{c}_\xi(x)=(s^*_{M\Sigma}\widetilde{c}_{\Sigma\mathbb{P}})(x)=\widetilde{c}_{\Sigma\mathbb{P}}\circ s_{M\Sigma}=\widetilde{c}_{\Sigma\mathbb{P}}(x,\,\xi)\,, \tag{47}$$

ensures the coincidence of images of sections $\widetilde{c}_\xi(x):M\to\mathbb{P}$ and $\widetilde{c}_{\Sigma\mathbb{P}}(x,\,\xi):\Sigma\to\mathbb{P}$, respectively. With the aid of the above results, we arrive at the useful result

$$\widetilde{c}_{\Sigma\mathbb{P}}(x,\,\xi)=\widetilde{c}_{M\mathbb{P}}(x)\cdot c(\xi)\,. \tag{48}$$

A. Nonlinear Realizations and the Generalized Gauge Transformation

The generalized gauge transformation law is obtained by comparing bundle elements $p \in \mathbb{P}$ that differ by the left action of elements of the principal group G, $L_{g\in G}$. An arbitrary element $p \in \mathbb{P}$ can be written in terms of the null section with the aid of (45), (46) and (48) as

$$p = s_{M\mathbb{P}}(x) = R_a \circ \widetilde{c}_{\Sigma\mathbb{P}}(x,\, \xi)\,,\; a \in H. \tag{49}$$

Performing a gauge transformation on p we obtain the orbit $\lambda(p)$ defining a curve through $(x,\, \xi)$ in Σ

$$\lambda(p) = L_{g(x)} \circ p = R_{a'} \circ \widetilde{c}_{\Sigma\mathbb{P}}(x,\, \xi')\,;\;\; g(x) \in G,\;\; a' \in H. \tag{50}$$

Comparison of (49) with (50) leads to

$$L_{g(x)} \circ R_a \circ \widetilde{c}_{\Sigma\mathbb{P}}(x,\, \xi) = R_{a'} \circ \widetilde{c}_{\Sigma\mathbb{P}}(x,\, \xi')\,. \tag{51}$$

By virtue of the commutability [37] of left and right group translations of elements belonging to G, i.e. $L_g \circ R_h = R_h \circ L_g$, (51) may be recast as

$$L_{g(x)} \circ \widetilde{c}_{\Sigma\mathbb{P}}(x,\, \xi) = R_h \circ \widetilde{c}_{\Sigma\mathbb{P}}(x,\, \xi')\,. \tag{52}$$

where $R_{a^{-1}} \circ R_{a'} \equiv R_{a'a^{-1}} := R_h$ and $a'a^{-1} \equiv h \in H$. Equation (52) constitute a generalized gauge transformation. Performing the pullback of (52) with respect to the section $s_{M\Sigma}$ leads to

$$L_{g(x)} \circ \widetilde{c}_{\xi}(x) = R_{h(\xi,\, g(x))} \circ \widetilde{c}_{\xi'}(x)\,. \tag{53}$$

Thus, the left action L_g of G is a map that acts on $\mathbb{P}$ and Σ. In particular, L_g acting on fibers defined as orbits of the right action describes diffeomorphisms that transforming fibers over $\widetilde{c}_{\xi}(x)$ into the fibers $\widetilde{c}_{\xi'}(x)$ of Σ while simultaneously being displaced along H fibers via the action of R_h. Equation (53) states that nonlinear realizations of G mod H is determined by the action of an arbitrary element $g \in G$ on the quotient space G/H transforming one coset into another as

$$L_g : G/H \to G/H,\;\; c(\xi) \to c(\xi') \tag{54}$$

inducing a diffeomorphism $\xi \to \xi'$ on G/H. To simplify the action induced by (53) for calculation purposes we proceed as follows. Departing from (47) and substituting $s_{M\Sigma} = R_c \circ \widetilde{c}_{M\mathbb{P}}$ we get

$$\widetilde{c}_{\xi}(x) = \widetilde{c}_{\Sigma\mathbb{P}} \circ R_c \circ \widetilde{c}_{M\Sigma}. \tag{55}$$

Using $\widetilde{c}_{M\mathbb{P}} \circ R_c = R_c \circ \widetilde{c}_{M\mathbb{P}}$, (55) becomes $\widetilde{c}_{\xi}(x) = R_c \circ \widetilde{c}_{\Sigma\mathbb{P}} \circ \widetilde{c}_{M\Sigma} = R_c \circ \widetilde{c}_{M\mathbb{P}}$, where the last equality follows from use of $\widetilde{c}_{M\mathbb{P}} = \widetilde{c}_{\Sigma\mathbb{P}} \circ \widetilde{c}_{M\Sigma}$. By way of analogy, we assume $\widetilde{c}_{\xi'}(x) \equiv R_{c'} \circ \widetilde{c}_{M\mathbb{P}}$. Upon substitution of $\widetilde{c}_{\xi'}$ into (53) we obtain

$$L_g \circ R_c \circ \widetilde{c}_{M\mathbb{P}} = R_{h(\xi,\, g(x))} \circ R_{c'} \circ \widetilde{c}_{M\mathbb{P}}, \tag{56}$$

which after implementing the group actions is equivalent to,

$$g \cdot \widetilde{c}_{M\mathbb{P}} \cdot c = \widetilde{c}_{M\mathbb{P}} \cdot c' \cdot h. \tag{57}$$

Operating on (57) from the left by $\widetilde{c}_{M\mathbb{P}}^{-1}$ and making use of $g = \widetilde{c}_{M\mathbb{P}}^{-1} g \widetilde{c}_{M\mathbb{P}}$, we get $\left(\widetilde{c}_{M\mathbb{P}}^{-1} \cdot g \cdot \widetilde{c}_{M\mathbb{P}}\right) \cdot c = c' \cdot h$ which leads to $g \cdot c_{\xi} = c_{\xi'} \cdot h$, or

$$c' = g \cdot c \cdot h^{-1} \tag{58}$$

in short, where $c \equiv c_{\xi}$ and $c' \equiv c_{\xi'}$. Observe that the element h is a function whose argument is the couple $(\xi,\, g(x))$. The transformation rule (58) is in fact the key equation to determine the nonlinear realizations of G and specifies a unique H-valued field $h(\xi,\, g(x))$ on G/H.

Consider a family of sections $\{\widehat{c}(x,\, \xi)\}$ defined [34] on Σ by

$$\widehat{c}(x,\, \xi) := c \circ \widetilde{c}(x,\, \xi) = c\,(\widetilde{c}(x,\, \xi))\,. \tag{59}$$

Taking $\Pi_{\mathbb{P}\Sigma} \circ R_h \circ\, \widetilde{c}_{\Sigma\mathbb{P}} = \Pi_{\mathbb{P}\Sigma} \circ \widetilde{c}_{\Sigma\mathbb{P}} = (id)_{\Sigma}$ into account, we can explicitly exhibit the fact that the left action L_g of G on the null sections $\widetilde{c}_{\Sigma\mathbb{P}} : \mathbb{P} \to \Sigma$ induces an equivalence relation between differing elements $\widetilde{c}_{\xi},\, \widetilde{c}_{\xi'} \in \Sigma$ given by

$$\Pi_{\mathbb{P}\Sigma} \circ L_g \circ \widehat{c}_{\xi} = \Pi_{\mathbb{P}\Sigma} \circ R_{h(\xi,\, g(x))} \circ \widehat{c}_{\xi'} = R_{h(\xi,\, g(x))} \circ \widetilde{c}_{\xi'}, \tag{60}$$

so that

$$\widetilde{c}'_{\xi} := R_{h(\xi,\, g(x))} \circ \widetilde{c}_{\xi'} = L_g \circ \widetilde{c}_{\xi}. \tag{61}$$

From (61) we can write

$$\widetilde{c}_{\xi} \xmapsto{L_g} \widetilde{c}'_{\xi} = R_{h(\xi,\, g(x))} \circ \widetilde{c}_{\xi'} \ \forall h \in H. \tag{62}$$

Equation (62) gives rise to a complete partition of G/H into equivalence classes $\Pi^{-1}_{\mathbb{P}\Sigma}(\xi)$ of left cosets [34, 38]

$$cH = \left\{ R_{h(\xi,\, g(x))} \circ c / c \in G/H, \forall h \in H \right\} = \left\{ ch_1,\, ch_2, ..., \, ch_n \right\}, \tag{63}$$

where $c \in (G - H)$ plays the role of the fibers attached to each point of Σ. The elements ch_i are single representatives of each equivalence class $R_{h(\xi,\, g(x))} \circ c = cH \in \widetilde{\pi}^{-1}_{\Sigma M}(\mathcal{U})$. Thus, any diffeomorphism $L_g \circ \widetilde{c}_{\xi}$ on Σ together with the H-valued function $h(\xi,\, g(x))$ determine a unique gauge transformation $\widetilde{c}'_{\xi} = R_{h(\xi,\, g(x))} \circ \widetilde{c}_{\xi'}$. This demonstrates that gauge transformations are those diffeomorphisms on Σ that map fibers over $c(\xi)$ into fibers over $c(\xi')$ and simultaneously preserves the action of H.

IV. COVARIANT COSET FIELD TRANSFORMATIONS

We now proceed to determine the transformation behavior of parameters belonging to G/H. The elements of the CA and Lorentz groups are respectively parameterized about the identity element as

$$g = e^{i\epsilon^{\alpha}\mathbf{P}_{\alpha}} e^{i\alpha^{\mu\nu}\ {}^{\dagger}\mathbf{S}_{\mu\nu}} e^{i\beta^{\mu\nu}\mathbf{L}_{\mu\nu}} e^{ib^{\alpha}\mathbf{\Delta}_{\alpha}} e^{i\varphi\mathbf{D}},\ h = e^{iu^{\mu\nu}\mathbf{L}_{\mu\nu}}. \tag{64}$$

Elements of the coset space G/H are coordinatized by

$$c = e^{-i\xi^{\alpha}\mathbf{P}_{\alpha}} e^{ih^{\mu\nu}\ {}^{\dagger}\mathbf{S}_{\mu\nu}} e^{i\zeta^{\alpha}\mathbf{\Delta}_{\alpha}} e^{i\phi\mathbf{D}}. \tag{65}$$

We consider transformations with infinitesimal group parameters ϵ^{α}, $\alpha^{\mu\nu}$, $\beta^{\mu\nu}$, b^{α} and φ. The transformed coset parameters read $\xi'^{\alpha} = \xi^{\alpha} + \delta\xi^{\alpha}$, $h'^{\mu\nu} = h^{\mu\nu} + \delta h^{\mu\nu}$, $\zeta'^{\alpha} = \zeta^{\alpha} + \delta\zeta^{\alpha}$ and $\phi' = \phi + \delta\phi$. Note that $u^{\mu\nu}$ is infinitesimal. The translational coset field variations reads

$$\delta\xi^{\alpha} = -\left(\alpha_{\beta}{}^{\alpha} + \beta_{\beta}{}^{\alpha}\right)\xi^{\beta} - \epsilon^{\alpha} - \varphi\xi^{\alpha} - \left[\left|\xi\right|^2 b^{\alpha} - 2\left(b \cdot \xi\right)\xi^{\alpha}\right]. \tag{66}$$

For the dilatons we get,

$$\delta\phi = \varphi + 2\left(b \cdot \xi\right) - \left\{u^{\alpha}_{\ \beta}\xi^{\beta} + \epsilon^{\alpha} + \varphi\xi^{\alpha} + \left[b^{\alpha}\left|\xi\right|^2 - 2\left(b \cdot \xi\right)\xi^{\alpha}\right]\right\}\partial_{\alpha}\phi. \tag{67}$$

Similarly for the special conformal 4-boosts we find,

$$\begin{aligned} \delta\zeta^{\alpha} \ &= \ u^{\alpha}_{\ \beta}\zeta^{\beta} + b^{\alpha} - \varphi\zeta^{\alpha} + 2\left[\left(b \cdot \xi\right)\zeta^{\alpha} - \left(b \cdot \zeta\right)\xi^{\alpha}\right] + \\ &-\left\{u^{\beta}_{\ \lambda}\xi^{\lambda} + \epsilon^{\beta} + \varphi\xi^{\beta} + \left[b^{\beta}\left|\xi\right|^2 - 2\left(b \cdot \xi\right)\xi^{\beta}\right]\right\}\partial_{\beta}\zeta^{\alpha}. \end{aligned} \tag{68}$$

Observe the homogeneous part of the special conformal coset parameter ζ^{α} has the same structure as that of the translational parameter ξ^{α} (with the substitutions: $\zeta^{\alpha} \to -\xi^{\alpha}$ and $-\epsilon^{\alpha} \to b^{\alpha}$). For the shear parameters we obtain

$$\delta r^{\alpha\beta} = \left(\alpha^{\gamma\alpha} + \beta^{\gamma\alpha}\right) r_{\gamma}{}^{\beta} + u^{\beta}_{\ \gamma} r^{\alpha\gamma} + 2b^{[\alpha}\xi^{\rho]} r_{\rho}{}^{\beta}, \tag{69}$$

where $r^{\alpha\beta} := e^{h^{\alpha\beta}}$. From $\delta r^{\alpha\beta}$ we obtain the nonlinear Lorentz transformation

$$u^{\alpha\beta} = \beta^{\alpha\beta} + 2b^{[\alpha}\xi^{\beta]} - \alpha^{\mu\nu}\tanh\left\{\frac{1}{2}\ln\left[r^{\alpha}_{\ \mu}\left(r^{-1}\right)^{\beta}_{\ \nu}\right]\right\}. \tag{70}$$

In the limit of vanishing special conformal 4-boost, this result coincides with that of Pinto *et al.* [30]. For vanishing shear, the result of Julve *et al* [31] is obtained.

In this section, all covariant coset field transformations were determined directly from the nonlinear transformation law (58). We observe that the translational coset parameter transforms as a coordinate under the action of G. From the shear coset variation, the explicit form of the nonlinear Lorentz-like transformation was obtained. From (70) it is clear that $u^{\alpha\beta}$ contains the linear Lorentz parameter in addition to conformal and shear contributions via the nonlinear 4-boosts and symmetric GL_4 parameters.

V. DECOMPOSITION OF CONNECTIONS IN $\pi_{\mathbb{P}M}:\mathbb{P}\to M$ INTO COMPONENTS IN $\pi_{\mathbb{P}\Sigma}:\mathbb{P}\to\Sigma$ AND $\pi_{\Sigma M}:\Sigma\to M$

Depending on which bundle is considered, either the total bundle $\mathbb{P}\to M$ or the intermediate bundles $\mathbb{P}\to\Sigma$, $\Sigma\to M$, we may construct corresponding Ehresmann connections for the respective space. With respect to M, we have the connection form

$$\omega=\widetilde{g}^{-1}\left(d+\pi^{*}_{\mathbb{P}M}\Omega_{M}\right)\widetilde{g}. \tag{71}$$

The gauge potential Ω_M is defined in the standard manner as the pullback of the connection ω by the null section $\widetilde{c}_{M\mathbb{P}}$, $\Omega_M=\widetilde{c}^{*}_{M\mathbb{P}}\omega\in T^{*}(M)$. With regard to the space Σ an alternative form of the connection is given by

$$\omega=a^{-1}\left(d+\pi^{*}_{\mathbb{P}\Sigma}\Gamma_{\Sigma}\right)a, \tag{72}$$

where the connection on Σ reads $\Gamma_\Sigma=\widetilde{c}^{*}_{\Sigma\mathbb{P}}\omega$. Carrying out a similar analysis and evaluating the tangent vector $X\in T_p(\Sigma)$ at each point ξ along the curve c_ξ on the coset space G/H that coincides with the section $\widetilde{c}^{*}_{\Sigma\mathbb{P}}$, we find the gauge transformation law

$$\omega\to\omega'=ad_{h^{-1}}\left(d+\omega\right). \tag{73}$$

Comparison of (71) and 72 leads to $\pi^{*}_{\mathbb{P}\Sigma}\Gamma_\Sigma=c^{-1}\left(d+\pi^{*}_{\mathbb{P}M}\Omega_M\right)c$. Taking account of $\widetilde{c}^{*}_{\Sigma\mathbb{P}}\Pi^{*}_{\mathbb{P}\Sigma}=(id)_{T^{*}(\Sigma)}$ which follows from $\Pi_{\mathbb{P}\Sigma}\circ\widetilde{c}_{\Sigma\mathbb{P}}=(id)_\Sigma$, we deduce

$$\Gamma_\Sigma=\widetilde{c}^{*}_{\Sigma\mathbb{P}}\left[c^{-1}\left(d+\pi^{*}_{\mathbb{P}M}\Omega_M\right)c\right]. \tag{74}$$

By use of the family of sections pulled back to Σ introduced in (59) we find $\widetilde{c}^{*}_{\Sigma\mathbb{P}}\left(c^{-1}dc\right)=\widehat{c}^{\,-1}d\widehat{c}$ and $\widetilde{c}^{*}_{\Sigma\mathbb{P}}R^{*}_{c}=R^{*}_{\widehat{c}}\widetilde{c}^{*}_{\Sigma\mathbb{P}}$. Recalling $\widetilde{\pi}^{*}_{\mathbb{P}M}=\widetilde{\pi}^{*}_{\mathbb{P}\Sigma}\widetilde{\pi}^{*}_{\Sigma M}$, we get $c^{-1}\widetilde{\pi}^{*}_{\mathbb{P}M}\Omega_M c=R^{*}_{c}\widetilde{\pi}^{*}_{\mathbb{P}M}\Omega_M$. With these results in hand, we obtain the alternative form of the connection Γ_Σ,

$$\Gamma_\Sigma=\widehat{c}^{-1}\left(d+\pi^{*}_{\Sigma M}\Omega_M\right)\widehat{c}. \tag{75}$$

Completing the pullback of Γ_Σ to M by means of $\widetilde{c}_{M\Sigma}$ we obtain, $\Gamma_M=\widetilde{c}^{*}_{M\Sigma}\Gamma_\Sigma$. By use of $\Gamma_\Sigma=\widetilde{c}^{*}_{\Sigma\mathbb{P}}\omega$ and (47) we find $\Gamma_M=s^{*}_{M\Sigma}\widetilde{c}^{*}_{\Sigma\mathbb{P}}\omega=\widetilde{c}^{*}_{\xi}\omega$. In terms of the substitution $\widehat{c}(x,\xi)\to\overline{c}(x)$ where $\overline{c}(x)$ is the pullback of $\widehat{c}(x,\xi)$ to M defined as $\overline{c}(x)=s^{*}_{M\Sigma}\widehat{c}=c\left(\widetilde{c}_\xi(x)\right)$, we arrive at the desired result

$$\mathbf{\Gamma}\equiv\Gamma_M=\overline{c}^{-1}\left(d+\Omega_M\right)\overline{c}, \tag{76}$$

which explicitly relates the connection $\mathbf{\Gamma}$ on Σ pulled back to M to its counterpart Ω_M.

The gauge transformation behavior of $\mathbf{\Gamma}$ may be determined directly by use of (29) and the transformation $\widetilde{c}'=g\widetilde{c}h^{-1}$. We calculate

$$\mathbf{\Gamma}'=h\widetilde{c}^{-1}g^{-1}d\left(g\widetilde{c}h^{-1}\right)+h\widetilde{c}^{-1}\Omega\widetilde{c}h^{-1}+h\widetilde{c}^{-1}\left(dg^{-1}\right)g\widetilde{c}h^{-1}. \tag{77}$$

Observing however, that

$$h\widetilde{c}^{-1}g^{-1}d\left(g\widetilde{c}h^{-1}\right)=h\widetilde{c}^{-1}\left(g^{-1}dg\right)\widetilde{c}h^{-1}+h\widetilde{c}^{-1}d\widetilde{c}h^{-1}+hdh^{-1}, \tag{78}$$

we obtain

$$\mathbf{\Gamma}'=h\left[\widetilde{c}^{-1}\left(d+\Omega\right)\widetilde{c}\right]h^{-1}+hdh^{-1}+h\widetilde{c}^{-1}d\left(gg^{-1}\right)\widetilde{c}h^{-1}. \tag{79}$$

Thus, we arrive at the gauge transformation law

$$\mathbf{\Gamma}'=h\mathbf{\Gamma}h^{-1}+hdh^{-1}. \tag{80}$$

According to the Lie algebra decomposition of $\mathfrak{g}$ into $\mathfrak{h}$ and $\mathfrak{c}$, the connection Γ_Σ can be divided into $\mathbf{\Gamma}_H$ defined on the subgroup H and $\mathbf{\Gamma}_{G/H}$ defined on G/H. From the transformation law (80) it is clear that $\mathbf{\Gamma}_H$ transforms inhomogeneously

$$\mathbf{\Gamma}'_H=h\mathbf{\Gamma}_H h^{-1}+hdh^{-1}, \tag{81}$$

while $\Gamma_{G/H}$ transforms as a tensor

$$\mathbf{\Gamma}'_{G/H} = h\mathbf{\Gamma}_{G/H}h^{-1}. \tag{82}$$

In this regard, only Γ_H transforms as a true connection. We use the gauge potential $\mathbf{\Gamma}$ to define the gauge covariant derivative

$$\nabla := (d + \rho(\mathbf{\Gamma})) \tag{83}$$

acting on ψ as $\nabla\psi = (d + \rho(\Gamma))\psi$ with the desired transformation property

$$(\nabla\psi(c(\xi)))' = \rho(h(\xi, g))\nabla\psi(c(\xi)) \simeq (1 + iu(\xi, g)\rho(H))\nabla\psi(c(\xi)) \tag{84}$$

leading to

$$\delta(\nabla\psi(c(\xi))) = iu(\xi, g)\rho(H)\nabla\psi(c(\xi)). \tag{85}$$

A. Conform-Affine Nonlinear Gauge Potential in $\pi_{\mathbb{P}M} : \mathbb{P} \to M$

The ordinary gauge potential defined on the total base space M reads

$$\Omega = -i\left(\overset{\mathrm{T}}{\Gamma}{}^{\alpha}\mathbf{P}_{\alpha} + \overset{\mathrm{C}}{\Gamma}{}^{\alpha}\mathbf{\Delta}_{\alpha} + \overset{\mathrm{D}}{\Gamma}\mathbf{D} + \overset{\mathrm{GL}}{\Gamma}{}^{\alpha\beta}\ {}^{\dagger}\mathbf{\Lambda}_{\alpha\beta}\right). \tag{86}$$

The horizontal basis vectors that span the horizontal tangent space $\mathbb{H}(\mathbb{P})$ of $\pi_{\mathbb{P}M} : \mathbb{P} \to M$ are given by

$$E_i = \widetilde{c}_{M\mathbb{P}*}\partial_i - \Omega_i. \tag{87}$$

The explicit form of the connections (86) are given by

$$\omega = -i\left[V^{\mu}_{M}\widetilde{\chi}_{\mu}{}^{\nu}\mathbf{P}_{\nu} - i\left(i\overline{\Theta}^{\alpha\beta}_{({}^{\dagger}\mathbf{\Lambda})} + \widetilde{\pi}^{*}_{\mathbb{P}M}\overset{\mathrm{GL}}{\Gamma}{}^{\alpha\beta}\right)\widetilde{\chi}_{\alpha}{}^{\nu}\widetilde{\chi}_{\beta}{}^{\nu}\ {}^{\dagger}\mathbf{\Lambda}_{\mu\nu} + \vartheta^{\mu}_{M}\widetilde{\beta}_{\mu}{}^{\nu}\mathbf{\Delta}_{\nu} - i\widetilde{\pi}^{*}_{\mathbb{P}M}\Phi_{M}\mathbf{D}\right] \tag{88}$$

where $\overline{\Theta}^{\alpha\beta}_{({}^{\dagger}\Lambda)} = \overline{\Theta}^{\alpha\beta}_{(\mathbf{L})} + \overline{\Theta}^{\alpha\beta}_{(\mathrm{SY})}$, with right invariant Maurer-Cartan forms

$$\overline{\Theta}^{\mu\nu}_{(\mathbf{L})} = i\widetilde{\beta}^{[\nu|}{}_{\gamma}d\widetilde{\beta}^{|\mu]\gamma} - 2idb^{\mu}\epsilon^{\nu} \text{ and } \overline{\Theta}^{\mu\nu}_{(\mathrm{SY})} = i\widetilde{\alpha}^{(\nu|}{}_{\gamma}d\widetilde{\alpha}^{|\mu)\gamma}. \tag{89}$$

The linear connection Ω_M varies under the action of G as

$$\delta\Omega = \Omega' - \Omega = \delta\overset{\mathrm{T}}{\Gamma}{}^{\mu}\mathbf{P}_{\mu} + \delta\overset{\mathrm{C}}{\Gamma}{}^{\mu}\mathbf{\Delta}_{\mu} + \delta\overset{\mathrm{D}}{\Gamma}\mathbf{D} + \delta\overset{\mathrm{GL}}{\Gamma}{}^{\beta\nu}\ {}^{\dagger}\mathbf{\Lambda}_{\beta\nu} \tag{90}$$

where

$$\begin{aligned}
\delta\overset{\mathrm{T}}{\Gamma}{}^{\mu} &= {}^{\dagger}\overset{\mathrm{GL}}{D}\epsilon^{\mu} - \overset{\mathrm{T}}{\Gamma}{}^{\alpha}\left(\alpha_{\alpha}{}^{\mu} + \beta_{\alpha}{}^{\mu} + \varphi\delta_{\alpha}{}^{\mu}\right) - \overset{\mathrm{D}}{\Gamma}\epsilon^{\mu},\\
\delta\overset{\mathrm{C}}{\Gamma}{}^{\mu} &= {}^{\dagger}\overset{\mathrm{GL}}{D}b^{\mu} - \overset{\mathrm{C}}{\Gamma}{}^{\alpha}\left(\alpha_{\alpha}{}^{\mu} + \beta_{\alpha}{}^{\mu} - \varphi\delta_{\alpha}{}^{\mu}\right) + \overset{\mathrm{D}}{\Gamma}b^{\mu},\\
\delta\overset{\mathrm{GL}}{\Gamma}{}^{\alpha\beta} &= {}^{\dagger}\overset{\mathrm{GL}}{D}\left(\alpha^{\alpha\beta} + \beta^{\alpha\beta}\right) + \left(\overset{\mathrm{T}}{\Gamma}{}^{[\alpha}b^{\beta]} + \overset{\mathrm{C}}{\Gamma}{}^{[\alpha}\epsilon^{\beta]}\right),\\
\delta\overset{\mathrm{D}}{\Gamma} &= d\varphi + 2\left(\overset{\mathrm{C}}{\Gamma}{}^{\alpha}\epsilon_{\alpha} - \overset{\mathrm{T}}{\Gamma}{}^{\alpha}b_{\alpha}\right).
\end{aligned} \tag{91}$$

The components of $\overline{\omega}$ on M are identified as spacetime quantities and are determined from the pullback of the corresponding (quotient space) quantities defined on Σ:

$$V^{\mu}_{M} = s^{*}_{M\Sigma}V^{\mu}_{\Sigma},\ \vartheta^{\mu}_{M} = s^{*}_{M\Sigma}\vartheta^{\mu}_{\Sigma},\ \Phi_{M} = s^{*}_{M\Sigma}\Phi_{\Sigma} \text{ and } \Gamma^{\mu\nu}_{M} = s^{*}_{M\Sigma}\Gamma^{\mu\nu}_{\Sigma}. \tag{92}$$

In the following, we depart from the alternative form of the connection $\omega = a^{-1}\left(d + \Pi^{*}_{\mathbb{P}\Sigma}\Gamma_{\Sigma}\right)a$, $\forall\ a \in H$ on Σ.

B. Conform-Affine Nonlinear Gauge Potential in $\pi_{\mathbb{P}\Sigma}:\mathbb{P}\rightarrow\Sigma$

The components of ω in $\mathbb{P}\rightarrow\Sigma$ are oriented along the Lie algebra basis of H

$$\overset{\mathbf{L}}{\omega}=a^{-1}\left(d+i\widetilde{\pi}_{\mathbb{P}\Sigma}^{*}\overset{\circ}{\Gamma}{}^{\alpha\beta}\mathbf{L}_{\alpha\beta}\right)a=-i\overset{\mathbf{L}}{\omega}{}^{\alpha\beta}\mathbf{L}_{\alpha\beta}, \tag{93}$$

where

$$\overset{\mathbf{L}}{\omega}{}^{\alpha\beta}:=\left(i\overline{\Theta}_{(\mathbf{L})}^{\rho\sigma}+\widetilde{\pi}_{\mathbb{P}\Sigma}^{*}\Gamma_{[\mathbf{L}]}^{\rho\sigma}\right)\widetilde{\beta}_{[\rho}{}^{\alpha}\widetilde{\beta}_{\sigma]}{}^{\beta}. \tag{94}$$

C. Conform-Affine Nonlinear Gauge Potential on $\Pi_{\Sigma M}:\Sigma\rightarrow M$

The components of ω in $\Pi_{\Sigma M}:\Sigma\rightarrow M$ are oriented [33] along the Lie algebra basis of the quotient space G/H belonging to Σ

$$\overset{\mathbf{P}}{\omega}\;=\;-ia^{-1}\left(\widetilde{\pi}_{\Sigma M}^{*}V_{\Sigma}^{\nu}\mathbf{P}_{\nu}\right)a=-i\overset{\mathbf{P}}{\omega}{}^{\mu}\mathbf{P}_{\mu}, \tag{95}$$

$$\overset{\mathbf{\Delta}}{\omega}\;=\;-ia^{-1}\left(\widetilde{\pi}_{\Sigma M}^{*}\vartheta_{\Sigma}^{\nu}\mathbf{\Delta}_{\nu}\right)a=-i\overset{\mathbf{\Delta}}{\omega}{}^{\mu}\mathbf{\Delta}_{\mu}, \tag{96}$$

$$\overset{\mathbf{D}}{\omega}\;=\;-ia^{-1}\left(\widetilde{\pi}_{\Sigma M}^{*}\Phi_{\Sigma}\mathbf{D}\right)a=-i\omega_{[\mathbf{D}]}\mathbf{D}, \tag{97}$$

$$\overset{\mathrm{SY}}{\omega}\;=\;-ia^{-1}\left(\widetilde{\pi}_{\Sigma M}^{*}\Upsilon^{\alpha\beta}\mathbf{S}_{\alpha\beta}\right)a=-i\overset{\mathrm{SY}}{\omega}{}^{\alpha\beta}\mathbf{S}_{\alpha\beta}, \tag{98}$$

where

$$\overset{\mathbf{P}}{\omega}{}^{\mu}\;:=\widetilde{\pi}_{\Sigma M}^{*}V_{\Sigma}^{\nu}\widetilde{\beta}_{\nu}{}^{\mu},\;\overset{\mathbf{\Delta}}{\omega}{}^{\mu}:=\widetilde{\pi}_{\Sigma M}^{*}\vartheta_{\Sigma}^{\nu}\widetilde{\beta}_{\nu}{}^{\mu}, \tag{99}$$

$$\omega_{[\mathbf{D}]}\;:=\widetilde{\pi}_{\Sigma M}^{*}\Phi_{\Sigma},\;\overset{\mathrm{SY}}{\omega}{}^{\alpha\beta}:=\widetilde{\pi}_{\mathbb{P}\Sigma}^{*}\Upsilon^{\rho\sigma}\widetilde{\alpha}_{(\rho}{}^{\alpha}\widetilde{\alpha}_{\sigma)}{}^{\beta}. \tag{100}$$

By direct computation we obtain

$$\mathbf{\Gamma}_{\Sigma}^{\mathrm{CA}}=-i\left(V_{\Sigma}^{\mu}\mathbf{P}_{\mu}+i\vartheta_{\Sigma}^{\mu}\mathbf{\Delta}_{\mu}+\Phi_{\Sigma}\mathbf{D}+\Gamma_{\Sigma}^{\alpha\beta}\mathbf{\Lambda}_{\alpha\beta}\right). \tag{101}$$

The nonlinear translational and special conformal connection coefficients V_{Σ}^{ν} and ϑ_{Σ}^{ν} read

$$V_{\Sigma}^{\beta}=\widetilde{\pi}_{\Sigma M}^{*}\left[e^{\phi}\left(\upsilon^{\beta}\left(\xi\right)+r_{\ \sigma}^{\alpha}\overset{\mathrm{C}}{\Gamma}{}^{\sigma}\mathfrak{B}_{\alpha}{}^{\beta}\left(\xi\right)\right)\right], \tag{102}$$

$$\vartheta_{\Sigma}^{\beta}=\widetilde{\pi}_{\Sigma M}^{*}\left[e^{-\phi}\left(\upsilon^{\beta}\left(\zeta\right)+\upsilon^{\sigma}\left(\xi\right)\mathfrak{B}_{\sigma}{}^{\beta}\left(\zeta\right)\right)\right], \tag{103}$$

with

$$\upsilon_{i}^{\beta}\left(\xi\right):=r_{\sigma}^{\beta}\left({}^{\dagger}\overset{\mathrm{GL}}{D}_{i}\xi^{\sigma}+\overset{\mathrm{D}}{\Gamma}_{i}\xi^{\sigma}+\overset{\mathrm{T}}{\Gamma}{}^{\sigma}_{i}\right),\;\mathfrak{B}_{\alpha}{}^{\rho}\left(\xi\right):=\left(\left|\xi\right|^{2}\delta_{\alpha}{}^{\rho}-2\xi_{\alpha}\xi^{\rho}\right). \tag{104}$$

The nonlinear GL_4 and dilaton connections are given by

$$\Gamma_{\Sigma}^{\mu\nu}=\widehat{\Gamma}{}^{\mu\nu}+2\zeta^{[\mu}\varpi^{\nu]}, \tag{105}$$

$$\Phi=\widetilde{\pi}_{\Sigma M}^{*}\left(\zeta_{\beta}\varpi^{\beta}\right)-\frac{1}{2}d\phi, \tag{106}$$

with

$$\widehat{\Gamma}^{\mu\nu} := \widetilde{\pi}^*_{\Sigma M}\left[\left(r^{-1}\right)^{\mu}_{\ \sigma}\overset{\text{GL}}{\Gamma}{}^{\sigma\beta} r_{\beta}^{\ \nu} - \left(r^{-1}\right)^{\mu}_{\ \sigma} dr^{\sigma\nu}\right] \tag{107}$$

and

$$\varpi^{\nu} := \upsilon^{\nu} + r^{\nu}_{\ \alpha}\overset{\text{C}}{\Gamma}{}^{\alpha}. \tag{108}$$

The nonlinear GL_4 connection can be expanded in the GL_4 Lie algebra according to $\Gamma^{\alpha\beta}\ {}^{\dagger}\mathbf{\Lambda}_{\alpha\beta} = \overset{\circ}{\Gamma}{}^{\alpha\beta}\mathbf{L}_{\alpha\beta} + \Upsilon^{\alpha\beta}$ ${}^{\dagger}\mathbf{S}_{\alpha\beta}$, where

$$\overset{\circ}{\Gamma}{}^{\alpha\beta}_{\Sigma} := \widehat{\Gamma}^{[\alpha\beta]} + 2\zeta^{[\alpha}\varpi^{\beta]},\ \Upsilon^{\alpha\beta}_{\Sigma} := \widehat{\Gamma}^{(\alpha\beta)}. \tag{109}$$

The symmetric GL_4 (shear) gauge fields Υ are distortion fields describing the difference between the general linear connection and the Levi-Civita connection.

We define the (group) algebra bases e_{ν} and h_{ν} dual to the translational and special conformal 1-forms V^{μ} and ϑ^{μ} as

$$e_{\mu} \ : = e_{\mu}^{\ i} s_{M\Sigma*}\partial_i = \partial_{\xi^{\mu}} - e_{\mu}^{\ i}\widetilde{e}_i, \tag{110}$$

$$h_{\mu} \ : = h_{\mu}^{\ i} s_{M\Sigma*}\partial_i = \partial_{\zeta^{\mu}} - h_{\mu}^{\ i}\widetilde{h}_i, \tag{111}$$

with corresponding tetrad-like components

$$e_i^{\ \mu}(\xi) \ = \ e^{\phi}\left(\upsilon_i^{\ \mu}(\xi) + r^{\alpha}_{\ \sigma}\overset{\text{C}}{\Gamma}{}_i^{\ \sigma}\mathfrak{B}_{\alpha}^{\ \mu}(\xi)\right), \tag{112}$$

$$h_i^{\ \mu}(\xi,\,\zeta) \ = \ e^{-\phi}\left(\upsilon^{\mu}_{\rho}(\zeta) + \upsilon^{\sigma}_i(\xi)\,\mathfrak{B}_{\sigma}^{\ \mu}(\zeta)\right), \tag{113}$$

and basis vectors (on M)

$$\widetilde{e}_j(\xi) = \widetilde{c}_{M\Sigma*}\partial_j - e^{\phi}\left[r_{\mu}^{\ \nu}\left(\overset{\text{GL}}{\Gamma}{}_{j\alpha}^{\ \ \mu}\xi^{\alpha} + \overset{\text{D}}{\Gamma}_j\xi^{\mu} + \overset{\text{T}}{\Gamma}{}_j^{\ \mu}\right) + \overset{\text{C}}{\Gamma}{}_j^{\ \sigma} r^{\mu}_{\ \sigma}\mathfrak{B}_{\mu}^{\ \nu}(\xi)\right]\partial_{\xi^{\nu}} \tag{114}$$

and

$$\widetilde{h}_j(\xi,\,\zeta) = \widetilde{c}_{M\Sigma*}\partial_j + e^{-\phi}\left[r^{\mu}_{\ \rho}\left(\overset{\text{GL}}{\Gamma}{}_{j\alpha}^{\ \ \rho}\zeta^{\alpha} + \overset{\text{C}}{\Gamma}{}_j^{\ \rho}\right) + r^{\gamma}_{\ \sigma}\left(\overset{\text{GL}}{\Gamma}{}_{j\alpha}^{\ \ \sigma}\xi^{\alpha} + \overset{\text{D}}{\Gamma}_j\xi^{\sigma} + \overset{\text{T}}{\Gamma}{}_j^{\ \sigma}\right)\mathfrak{B}^{\mu}_{\gamma}(\zeta)\right]\partial_{\zeta^{\mu}}. \tag{115}$$

Here $\upsilon^{\beta}(\zeta) = \upsilon^{\beta}(\xi \to \zeta)$, $\mathfrak{B}^{\beta}_{\alpha}(\zeta) = \mathfrak{B}^{\rho}_{\alpha}(\xi \to \zeta)$. By definition, the basis vectors satisfy the orthogonality relations

$$\langle V^{\mu}_{\Sigma}|\widetilde{e}_j\rangle = 0,\ \left\langle \vartheta^{\mu}_{\Sigma}|\widetilde{h}_j\right\rangle = 0,\ \langle V^{\mu}|e_{\nu}\rangle = \delta^{\mu}_{\nu},\ \langle \vartheta^{\mu}|h_{\nu}\rangle = \delta^{\mu}_{\nu}. \tag{116}$$

We introduce the dilatonic and symmetric GL_4 *algebra* bases

$$\flat := \partial_{\phi} - d^i\widetilde{d}_i,\ \ f_{\mu\nu} := \partial_{\alpha^{\mu\nu}} - f_{\mu\nu}^{\ \ i}\widetilde{f}_i \tag{117}$$

with *auxiliary soldering* components d_i and $f_i^{\ \mu\nu}$,

$$d_i \ = \ \zeta_{\sigma} r^{\sigma}_{\ \rho}\left({}^{\dagger}\overset{\text{GL}}{D}_i\xi^{\rho} + \overset{\text{D}}{\Gamma}_i\xi^{\rho} + \overset{\text{T}}{\Gamma}{}_i^{\ \rho} + \overset{\text{C}}{\Gamma}{}_i^{\ \rho}\right) - \frac{1}{2}\partial_i\phi, \tag{118}$$

$$f_i^{\ \mu\nu} \ = \ \left(r^{-1}\right)^{\mu}_{\ \sigma}\overset{\text{GL}}{\Gamma}{}_i^{\ \sigma\beta} r_{\beta}^{\ \nu} - \left(r^{-1}\right)^{\mu}_{\ \sigma}\partial_i r^{\sigma\nu}. \tag{119}$$

The *coordinate* bases $\widetilde{d}_j$ and $\widetilde{f}_j$ read

$$\widetilde{d}_j(\xi,\,\zeta,\,\phi,\,h) := \widetilde{c}_{M\Sigma*}\partial_j - \zeta_{\sigma} r^{\sigma}_{\ \rho}\left({}^{\dagger}\overset{\text{GL}}{\Gamma}{}^{\ \rho}_{j\gamma}\xi^{\gamma} + \overset{\text{D}}{\Gamma}_j\xi^{\rho} + \overset{\text{T}}{\Gamma}{}_j^{\ \rho} + \overset{\text{C}}{\Gamma}{}_j^{\ \rho}\right)\partial_{\phi}, \tag{120}$$

and

$$\widetilde{f}_j\,(\xi,\,h) := \widetilde{c}_{M\Sigma*}\partial_j - \left(\left(r^{-1}\right)^{(\mu|}{}_{\sigma} \overset{\mathrm{GL}}{\Gamma}{}_{j}{}^{\sigma\beta} r_{\beta}{}^{|\nu)} - \left(r^{-1}\right)^{(\mu|}{}_{\sigma}\,\partial_j r^{\sigma|\nu)} \right) \partial_{h^{\mu\nu}}. \tag{121}$$

The bases satisfy

$$\left\langle \Phi|\widetilde{d}_i \right\rangle = 0,\ \left\langle \Upsilon^{\alpha\beta}|\widetilde{f}_i \right\rangle = 0,\ \langle \Phi|\flat\rangle = I,\ \left\langle \Upsilon^{\alpha\beta}|f_{\mu\nu}\right\rangle = \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}. \tag{122}$$

With the basis vectors and tetrad components in hand, we observe

$$\begin{aligned} V^{\mu}_{M} &:= dx^i \otimes e^{\mu}_{i},\ \vartheta^{\mu}_{M} := dx^i \otimes h^{\mu}_{i}, \\ \Phi_M &:= dx^i \otimes e^{\alpha}_{i}\,\langle \Phi|e_{\alpha}\rangle = dx^i \otimes d_i. \end{aligned} \tag{123}$$

The symmetric and antisymmetric GL_4 connection pulled back to M is given by

$$\begin{aligned} \Upsilon^{\mu\nu}_{M} &= dx^i \otimes e^{\alpha}_{i}\,\langle \Upsilon^{\mu\nu}_{\Sigma}|e_{\alpha}\rangle := dx^i \otimes f_i{}^{\mu\nu}, \\ \overset{\circ}{\Gamma}{}^{\mu\nu}_{M} &= dx^i \otimes e^{\alpha}_{i} \left\langle \overset{\circ}{\Gamma}{}^{\mu\nu}_{\Sigma}|e_{\alpha} \right\rangle := dx^i \otimes \overset{\circ}{\Gamma}{}^{\mu\nu}_{i}. \end{aligned} \tag{124}$$

With the aid of (123) and (124), we determine

$$V^{\beta}_{i} := e_i{}^{\alpha}\left\langle V^{\beta}_{\Sigma}|e_{\alpha}\right\rangle = e_i{}^{\alpha}\delta^{\beta}_{\alpha} = e_i{}^{\beta},\ \vartheta^{\beta}_{i} \equiv h^{\beta}_{i},\ \Upsilon^{\mu\nu}_{i} \equiv f_i{}^{\mu\nu},\ \Phi_i \equiv d_i. \tag{125}$$

The horizontal tangent subspace vectors in $\widetilde{\pi}_{\mathbb{P}\Sigma} : \mathbb{P} \to \Sigma$ are given by

$$\widehat{E}_i = \widetilde{c}_{M\mathbb{P}*}\widetilde{e}_i + i\widetilde{c}_{M\Sigma*}\left\langle \overset{\circ}{\Gamma}{}^{\alpha\beta}|\widetilde{e}_i \right\rangle \overset{\mathrm{Int}}{\widehat{\mathfrak{R}}}{}^{(\mathbf{L})}_{\alpha\beta}, \tag{126}$$

$$\widehat{E}_{\mu} = \widetilde{c}_{\Sigma\mathbb{P}*}\widetilde{e}_{\mu} + i\left\langle \overset{\circ}{\Gamma}{}^{\alpha\beta}|\widetilde{e}_{\mu} \right\rangle \overset{\mathrm{Int}}{\widehat{\mathfrak{R}}}{}^{(\mathbf{L})}_{\alpha\beta}, \tag{127}$$

and satisfy

$$\left\langle \overset{\mathbf{L}}{\omega}|\widehat{E}_j \right\rangle = 0 = \left\langle \overset{\mathbf{L}}{\omega}|\widehat{E}_{\mu} \right\rangle. \tag{128}$$

The right invariant fundamental vector operator appearing in (126) or (127) is given by

$$\widehat{\mathfrak{R}}{}^{(\mathbf{L})}_{\mu\nu} = i\left(\widetilde{\beta}_{[\mu|}{}^{\gamma}\frac{\partial}{\partial\widetilde{\beta}^{|\nu]\gamma}} + \epsilon_{[\mu}\frac{\partial}{\partial\epsilon^{\nu]}} \right). \tag{129}$$

On the other hand, the vertical tangent subspace vector in $\widetilde{\pi}_{\mathbb{P}\Sigma} : \mathbb{P} \to \Sigma$ satisfies

$$\left\langle \overset{\mathbf{L}}{\omega}|\widehat{\mathfrak{L}}{}^{(\mathbf{L})}_{\mu\nu} \right\rangle = \mathbf{L}_{\mu\nu} = \left\langle \overset{\mathbf{L}}{\omega}|\widehat{\mathfrak{R}}{}^{(\mathbf{L})}_{\mu\nu} \right\rangle, \tag{130}$$

where

$$\widehat{\mathfrak{L}}{}^{(\mathbf{L})}_{\mu\nu} = i\widetilde{\beta}_{\gamma[\mu|}\frac{\partial}{\partial\widetilde{\beta}_{\gamma}{}^{|\nu]}},\ \widehat{\mathfrak{R}}{}^{(\mathbf{L})}_{\mu\nu} = i\left(\widetilde{\beta}_{[\mu|}{}^{\gamma}\frac{\partial}{\partial\widetilde{\beta}^{|\nu]\gamma}} + \epsilon_{[\mu}\frac{\partial}{\partial\epsilon^{\nu]}} \right). \tag{131}$$

and $\widetilde{\beta}_{\mu}{}^{\nu} := e^{\beta_{\mu}{}^{\nu}} = \delta_{\mu}{}^{\nu} + \beta_{\mu}{}^{\nu} + \frac{1}{2!}\beta_{\mu}{}^{\gamma}\beta_{\gamma}{}^{\nu} + \cdots$. The horizontal tangent subspace vectors in $\Pi_{\Sigma M} : \Sigma \to M$ are given by

$$\widetilde{E}_j = \widetilde{c}_{\Sigma\mathbb{P}*}\widetilde{e}_j,\ \widetilde{H}_j = \widetilde{c}_{\Sigma\mathbb{P}*}\widetilde{h}_j,\ \widehat{E}{}^{(\mathbf{D})}_{i} = \widetilde{c}_{\Sigma\mathbb{P}*}\widetilde{d}_j,\ \breve{E}_j = \widetilde{c}_{\Sigma\mathbb{P}*}\widetilde{f}_j, \tag{132}$$

and satisfy

$$\left\langle \overset{\mathbf{P}}{\omega}|\widetilde{E}_j \right\rangle = 0,\ \left\langle \overset{\boldsymbol{\Delta}}{\omega}|\widetilde{H}_j \right\rangle = 0,\ \left\langle \overset{\mathrm{SY}}{\omega}|\breve{E}_j \right\rangle = 0,\ \left\langle \overset{\mathbf{D}}{\omega}|\widehat{E}{}^{(\mathbf{D})}_{i} \right\rangle = 0. \tag{133}$$

The vertical tangent subspace vectors in $\Pi_{\Sigma M} : \Sigma \rightarrow M$ are given by

$$\widetilde{E}_{\mu} = \widetilde{c}_{\Sigma\mathbb{P}*}\widehat{\mathfrak{L}}\,_{\mu}^{(\mathbf{P})},\ \breve{E}_{\alpha\beta} = \widetilde{c}_{\Sigma\mathbb{P}*}\widehat{\mathfrak{L}}\,_{\alpha\beta}^{(\mathrm{SY})},\ \widetilde{H}_{\mu} = \widetilde{c}_{\Sigma\mathbb{P}*}\widehat{\mathfrak{L}}\,_{\mu}^{(\mathbf{\Delta})},\ \widehat{E}\,^{(\mathbf{D})} = \widetilde{c}_{\Sigma\mathbb{P}*}\widehat{\mathfrak{L}}\,^{(\mathbf{D})}, \tag{134}$$

and satisfy

$$\left\langle \overset{\mathbf{P}}{\omega}|\widetilde{E}_{\mu}\right\rangle = \mathbf{P}_{\mu},\ \left\langle \overset{\mathbf{\Delta}}{\omega}|\widetilde{H}_{\mu}\right\rangle = \mathbf{\Delta}_{\mu},\ \left\langle \overset{\mathrm{SY}}{\omega}|\breve{E}_{\alpha\beta}\right\rangle = {}^{\dagger}\mathbf{S}_{\alpha\beta},\ \left\langle \overset{\mathbf{D}}{\omega}|\widehat{E}\,^{(\mathbf{D})}\right\rangle = \mathbf{D}. \tag{135}$$

The left invariant fundamental vector operators appearing in (134) are readily computed, the result being

$$\begin{gathered}\widehat{\mathfrak{L}}\,_{\mu}^{(\mathbf{P})} = i\widetilde{Q}^{\nu}{}_{\mu}\tfrac{\partial}{\partial\epsilon^{\nu}},\ \widehat{\mathfrak{L}}\,_{\mu}^{(\mathbf{\Delta})} = i\widetilde{W}^{\nu}{}_{\mu}\tfrac{\partial}{\partial b^{\nu}},\\ \widehat{\mathfrak{L}}\,_{\alpha\beta}^{(\mathrm{SY})} = i\widetilde{\alpha}_{\gamma(\mu|}\tfrac{\partial}{\partial\widetilde{\alpha}_{\gamma}{}^{|\nu)}},\ \widehat{\mathfrak{L}}\,^{(\mathbf{D})} = -i\epsilon^{\beta}\tfrac{\partial}{\partial\epsilon^{\beta}},\end{gathered} \tag{136}$$

where $\widetilde{\alpha}_{\mu}^{\ \nu} := e^{\alpha_{\mu}^{\ \nu}} = \alpha_{\mu}^{\ \nu} + \alpha_{\mu}^{\ \nu} + \frac{1}{2!}\alpha_{\mu}^{\ \gamma}\alpha_{\gamma}^{\ \nu} + \cdots$, $\widetilde{Q}_{\sigma}^{\ \alpha} := (\widetilde{\chi}_{\sigma}^{\ \alpha} + \delta_{\sigma}^{\ \alpha}e^{\varphi})$, $\widetilde{W}_{\sigma}^{\ \alpha} := (\widetilde{\chi}_{\sigma}^{\ \alpha} + \delta_{\sigma}^{\ \alpha}e^{-\varphi})$ satisfying $\left(\widetilde{Q}^{-1}\right)_{\sigma}^{\ \alpha} = \widetilde{Q}^{\alpha}_{\ \sigma}$ and $\left(\widetilde{W}^{-1}\right)_{\sigma}^{\ \alpha} = \widetilde{W}^{\alpha}_{\ \sigma}$. Making use of the transformation law of the nonlinear connection (80) we obtain

$$\delta\Gamma = \delta V^{\alpha}\mathbf{P}_{\alpha} + \delta\vartheta^{\alpha}\mathbf{\Delta}_{\alpha} + 2\delta\Phi\mathbf{D} + \delta\mathbf{\Gamma}^{\alpha\beta}\ {}^{\dagger}\mathbf{\Lambda}_{\alpha\beta} \tag{137}$$

where

$$\delta V^{\nu} = u_{\alpha}^{\ \ \nu}V^{\alpha},\ \delta\vartheta^{\nu} = u_{\alpha}^{\ \ \nu}\vartheta^{\alpha},\ \delta\Phi = 0,\ \delta\mathbf{\Gamma}^{\alpha\beta} = {}^{\dagger}\overset{\mathrm{GL}}{\nabla}u^{\alpha\beta}. \tag{138}$$

From $\delta\mathbf{\Gamma}^{\alpha\beta} = {}^{\dagger}\overset{\mathrm{GL}}{\nabla}u^{\alpha\beta}$ we observe that

$$\delta\Gamma^{[\alpha\beta]} = \overset{\circ}{\nabla}u^{\alpha\beta},\ \delta\Upsilon_{\alpha\beta} = 2u^{\rho}{}_{(\alpha|}\Upsilon_{\rho|\beta)}. \tag{139}$$

According to (138), the nonlinear translational and special conformal gauge fields transform as contravariant vector valued 1-forms under H, the antisymmetric part of $\mathbf{\Gamma}^{\alpha\beta}$ transforms inhomogeneously as a gauge potential and the nonlinear dilaton gauge field Φ transforms as a scalar valued 1-form. From (139) it is clear that the symmetric part of $\mathbf{\Gamma}^{\alpha\beta}$ is a tensor valued 1-form. Being 4-covectors we identify V^{ν} as coframe fields. The connection coefficient $\overset{\circ}{\Gamma}{}^{\alpha\beta}$ serves as the gravitational gauge potential. The remaining components of $\mathbf{\Gamma}$, namely ϑ, Υ and Φ are dynamical fields of the theory. As will be seen in the following subsection, the tetrad components of the coframe are used in conjunction with the H-metric to induce a spacetime metric on M.

VI. THE INDUCED METRIC

Since the Lorentz group H is a subgroup of G, we inherit the invariant ($\delta o_{\alpha\beta} = \delta o^{\alpha\beta} = 0$) (constant) metric of H, where $o^{\alpha\beta} = o_{\alpha\beta} = diag\,(-,\ +,\ +,\ +)$. With the aid of $o_{\alpha\beta}$ and the tetrad components $e_{i}^{\ \alpha}$ given in (112), we define the spacetime metric

$$g_{ij} = e_{i}^{\ \alpha}e_{j}^{\ \beta}o_{\alpha\beta}. \tag{140}$$

Observing ${}^{\dagger}\overset{\mathrm{GL}}{\nabla}o_{\alpha\beta} = -2\Upsilon_{\alpha\beta}$ (where we used $do_{\alpha\beta} = 0$) and taking account of the (second) transformation property (139), we interpret $\Upsilon_{\alpha\beta}$ as a sort of nonmetricity, i.e. a deformation (or distortion) gauge field that describes the difference between the general linear connection and the Levi-Civita connection of Riemannian geometry. In the limit of vanishing gravitational interactions $\overset{\mathrm{T}}{\Gamma}{}^{\sigma} \sim \overset{\mathrm{C}}{\Gamma}{}^{\sigma} \sim \overset{\circ}{\Gamma}{}^{\alpha}{}_{\beta} \sim \Upsilon^{\alpha}_{\ \beta} \sim \Phi \rightarrow 0$, $r^{\beta}_{\sigma} \rightarrow \delta^{\beta}_{\sigma}$ (to first order) and ${}^{\dagger}\overset{\mathrm{GL}}{D}\xi^{\sigma} \rightarrow d\xi^{\sigma}$. Under these conditions, the coframe reduces to $V^{\beta} \rightarrow e^{\phi}\delta^{\beta}_{\alpha}d\xi^{\alpha}$ leading to the spacetime metric

$$g_{ij} \rightarrow e^{2\phi}\delta^{\rho}_{\alpha}\delta^{\sigma}_{\beta}\left(\partial_{i}\xi^{\alpha}\right)\left(\partial_{j}\xi^{\beta}\right)o_{\rho\sigma} = e^{2\phi}\left(\partial_{i}\xi^{\alpha}\right)\left(\partial_{j}\xi^{\beta}\right)o_{\alpha\beta} \tag{141}$$

characteristic of a Weyl geometry.

VII. THE CARTAN STRUCTURE EQUATIONS

Using the nonlinear gauge potentials derived in (103), (105), (106), the covariant derivative defined on Σ pulled back to M has form

$$\nabla := d - iV^{\alpha}\mathbf{P}_{\alpha} - i\vartheta^{\alpha}\mathbf{\Delta}_{\alpha} - 2i\Phi\mathbf{D} - i\Gamma^{\alpha\beta\ \dagger}\mathbf{\Lambda}_{\alpha\beta}. \tag{142}$$

By use of (142) together with the relevant Lie algebra commutators we obtain the the bundle curvature

$$\mathbb{F} := \nabla\wedge\nabla = -i\mathcal{T}^{\alpha}\mathbf{P}_{\alpha} - i\mathcal{K}^{\alpha}\mathbf{\Delta}_{\alpha} - i\mathcal{Z}\mathbf{D} - i\mathbb{R}_{\alpha}{}^{\beta\ \dagger}\mathbf{\Lambda}^{\alpha}{}_{\beta}. \tag{143}$$

The field strength components of $\mathbb{F}$ are given by the first Cartan structure equations. They are respectively, the projectively deformed, Υ-distorted translational field strength

$$\mathcal{T}^{\alpha} := {}^{\dagger}\overset{\mathrm{GL}}{\nabla}V^{\alpha} + 2\Phi\wedge V^{\alpha}, \tag{144}$$

the projectively deformed, Υ-distorted special conformal field strength

$$\mathcal{K}^{\alpha} := {}^{\dagger}\overset{\mathrm{GL}}{\nabla}\vartheta^{\alpha} - 2\Phi\wedge\vartheta^{\alpha}, \tag{145}$$

the Ψ-deformed Weyl homothetic curvature 2-form (dilaton field strength)

$$\mathcal{Z} := d\Phi + \Psi,\ \Psi = V\cdot\vartheta - \vartheta\cdot V \tag{146}$$

and the general CA curvature

$$\mathbb{R}^{\alpha\beta} := \widehat{R}^{\ \alpha\beta} + \Psi^{\alpha\beta}, \tag{147}$$

with

$$\widehat{R}^{\ \alpha\beta} := \mathfrak{R}^{\alpha\beta} + \mathcal{R}^{\alpha\beta},\ \ \Psi^{\alpha\beta} := V^{[\alpha}\wedge\vartheta^{\beta]}. \tag{148}$$

Operator ${}^{\dagger}\overset{\mathrm{GL}}{\nabla}$ denotes the nonlinear covariant derivative built from volume preserving (VP) connection (i.e. excluding Φ) forms respectively. The Υ and $\overset{\circ}{\Gamma}$-affine curvatures in (148) read

$$\mathfrak{R}^{\alpha\beta}\ :\ = \overset{\circ}{\nabla}\Upsilon^{\alpha\beta} + \Upsilon^{\alpha}_{\gamma}\wedge\Upsilon^{\gamma\beta}, \tag{149}$$

$$\mathcal{R}^{\alpha\beta}\ :\ = d\overset{\circ}{\Gamma}{}^{\alpha\beta} + \overset{\circ}{\Gamma}{}_{\gamma}{}^{\alpha}\wedge\overset{\circ}{\Gamma}{}^{\gamma\beta}, \tag{150}$$

respectively. Operator $\overset{\circ}{\nabla}$ is defined with respect to the restricted connection $\overset{\circ}{\Gamma}{}^{\alpha\beta}$ given in (109).

The field strength components of the bundle curvature have the following group variations

$$\delta\mathbb{R}_{\alpha}^{\ \beta} = u_{\alpha}{}^{\gamma}\mathbb{R}^{\beta}{}_{\gamma} - u_{\gamma}{}^{\beta}\mathbb{R}_{\alpha}{}^{\gamma},\ \delta\mathcal{Z} = 0,\ \delta\mathcal{T}^{\alpha} = -u_{\beta}{}^{\alpha}\mathcal{T}^{\beta},\ \delta\mathcal{K}^{\alpha} = -u_{\beta}{}^{\alpha}\mathcal{K}^{\beta}. \tag{151}$$

A gauge field Lagrangian is built from polynomial combinations of the strength $\mathbb{F}$ defined as

$$\mathbb{F}\left(\Gamma\left(\Omega,\ D\xi\right),\ d\Gamma\right) := \nabla\wedge\nabla = d\Gamma + \Gamma\wedge\Gamma. \tag{152}$$

VIII. BIANCHI IDENTITIES

In what follows, the Bianchi identities (BI) play a central role. We therefore derive them presently.

1a) The 1^{st} translational BI reads,

$$\overset{\mathrm{GL}}{\nabla}\mathcal{T}^{a} = \widehat{R}^{\ \alpha}_{\ \ \beta}\wedge V^{\beta} + \Phi\wedge T^{a} + 2d\left(\Phi\wedge V^{\alpha}\right). \tag{153}$$

1b) Similarly to the case in (1a), the 1^{st} conformal BIs are respectively given by,

$$\overset{\mathrm{GL}}{\nabla}\mathcal{K}^{a} = \widehat{R}^{\ \alpha}_{\ \ \beta}\wedge\vartheta^{\beta} - \Phi\wedge\mathcal{K}^{a} - 2d\left(\Phi\wedge\vartheta^{\alpha}\right), \tag{154}$$

2a) The Υ and $\overset{\circ}{\Gamma}$-affine component of the 2^{nd} BI is given by

$$^{\dagger}\overset{\mathrm{GL}}{\nabla}\mathfrak{R}^{\alpha\beta} = 2\mathfrak{R}^{(\alpha|}{}_{\gamma}\Upsilon^{\gamma|\beta)},\ ^{\dagger}\overset{\mathrm{GL}}{\nabla}\mathcal{R}^{\alpha\beta} = 0, \tag{155}$$

respectively. Hence, the generalized 2^{nd} BI is given by

$$^{\dagger}\overset{\mathrm{GL}}{\nabla}\widehat{R}\ {}^{\alpha}{}_{\beta} = 2\mathfrak{R}^{(\alpha|}{}_{\gamma}\Upsilon^{\gamma|\rho)}o_{\rho\beta}. \tag{156}$$

Since the full curvature $\mathbb{R}^{\alpha\beta}$ is proportional to $\Psi^{\alpha\beta}$, it is necessary to consider

$$^{\dagger}\overset{\mathrm{GL}}{\nabla}\Psi^{\alpha\beta} = {}^{\dagger}\mathcal{T}^{\alpha}\wedge\vartheta^{\beta} + V^{\alpha}\wedge {}^{\dagger}\mathcal{K}^{\beta}, \tag{157}$$

from which we conclude

$$^{\dagger}\overset{\mathrm{GL}}{\nabla}\mathbb{R}^{\alpha\beta} = 2\mathfrak{R}^{(\alpha|}{}_{\gamma}\Upsilon^{\gamma|\beta)} + {}^{\dagger}\mathcal{T}^{\alpha}\wedge\vartheta^{\beta} + V^{\alpha}\wedge {}^{\dagger}\mathcal{K}^{\beta}. \tag{158}$$

2c) The dilatonic component of the 2^{nd} BI is given by

$$\overset{\mathrm{GL}}{\nabla}\mathcal{Z} = dZ + \overset{\mathrm{GL}}{\nabla}(V\wedge\vartheta) = \overset{\mathrm{GL}}{\nabla}\Psi + \Phi\wedge\Psi, \tag{159}$$

From the definition of Ψ, we obtain

$$\nabla\Psi = \mathcal{T}^{\alpha}\wedge\vartheta_{\alpha} + V_{\alpha}\wedge\mathcal{K}^{\alpha} + \Phi\wedge(V_{\alpha}\wedge\vartheta^{\alpha}). \tag{160}$$

Defining

$$\Sigma^{\mu\nu} := \mathbf{B}^{\mu\nu} + \Psi^{\mu\nu},\ \mathbf{B}^{\mu\nu} := B^{\mu\nu} + \mathcal{B}^{\mu\nu},\ B^{\mu\nu} := V^{\mu}\wedge V^{\nu},\ \ \mathcal{B}^{\mu\nu} := \vartheta^{\mu}\wedge\vartheta^{\nu}, \tag{161}$$

and asserting $V^{\alpha}\wedge\vartheta_{\alpha} = 0$, we find $\Sigma_{\mu\nu}\wedge\Sigma^{\mu\nu} = 0$. Using this result,we obtain

$$\nabla\Psi = \mathcal{T}^{\alpha}\wedge\vartheta_{\alpha} + V_{\alpha}\wedge\mathcal{K}^{\alpha}. \tag{162}$$

IX. ACTION FUNCTIONAL AND FIELD EQUATIONS

We seek an action for a local gauge theory based on the $CA(3,1)$ symmetry group. We consider the $3D$ topological invariants $\mathbb{Y}$ of the non-Riemannian manifold of CA connections. Our objective is the $4D$ boundary terms $\mathbb{B}$ obtained by means of exterior differentiation of these $3D$ invariants, i.e. $\mathbb{B} = d\mathbb{Y}$. The Lagrangian density of CA gravity is modeled after $\mathbb{B}$, with appropriate distribution of Lie star operators so as to re-introduce the dual frame fields. The generalized CA surface topological invariant reads

$$\mathbb{Y} = -\frac{1}{2l^2}\begin{bmatrix} \theta_{\mathcal{A}}\left(\mathcal{A}_a{}^b\wedge\widehat{R}\ {}_b{}^a + \frac{1}{3}\mathcal{A}_a{}^b\wedge\mathcal{A}_b{}^c\wedge\mathcal{A}_c{}^a\right) + \\ -\theta_{\mathcal{V}}\mathcal{V}_a\wedge\mathbf{T}^{\alpha} + \theta_{\Phi}\Phi\wedge\mathcal{Z} \end{bmatrix}, \tag{163}$$

where $\mathbf{T}^{\alpha} := \mathcal{T}^{\alpha} + \mathcal{K}^{\alpha}$. The associated total CA boundary term is given by,

$$\mathbb{B} = \frac{1}{2l^2}\begin{bmatrix} \widehat{R}_{\beta\alpha}\wedge\mathbf{B}^{\beta\alpha} + \Sigma^{[\beta\alpha]}\wedge\Sigma_{[\beta\alpha]} - \widehat{R}\ {}^{\alpha\beta}\wedge\widehat{R}_{\alpha\beta} - \mathcal{Z}\wedge\mathcal{Z} + \\ +\mathcal{K}_{\alpha}\wedge\mathcal{K}^{\alpha} + \mathcal{T}_{\alpha}\wedge\mathcal{T}^{\alpha} - \Phi\wedge(V_{\alpha}\wedge\mathcal{T}^{\alpha} + \vartheta_{\alpha}\wedge\mathcal{K}^{\alpha}) + \\ -\Upsilon_{\alpha\beta}\wedge\left(V^{\alpha}\wedge\mathcal{T}^{\beta} + \vartheta^{\alpha}\wedge\mathcal{K}^{\beta}\right). \end{bmatrix} \tag{164}$$

Using the boundary term (164) as a guide, we choose [48, 51, 54, 56, 66] an action of form

$$I = \int_{\mathcal{M}}\left\{\begin{array}{c} d(\mathcal{V}^{\alpha}\wedge\mathbf{T}_{\alpha}) + \widehat{R}\ {}^{\alpha\beta}\wedge\Sigma_{\star\alpha\beta} + \mathcal{B}_{\star\alpha\beta}\wedge\mathcal{B}^{\alpha\beta} + \Psi_{\star\alpha\beta}\wedge\Psi^{\alpha\beta} + \eta_{\star\alpha\beta}\wedge\eta^{\alpha\beta} \\ -\frac{1}{2}(\mathcal{R}_{\star\mu\nu}\wedge\mathcal{R}^{\mu\nu} + \mathcal{Z}\wedge\star\mathcal{Z}) + \mathcal{T}_{\star\alpha}\wedge\mathcal{T}^{\alpha} + \mathcal{K}_{\star\alpha}\wedge\mathcal{K}^{\alpha} + \\ -\Phi\wedge(\mathcal{T}^{\star\alpha}\wedge V_{\alpha} + \mathcal{K}^{\star\alpha}\wedge\vartheta_{\alpha}) - \Upsilon_{\alpha\beta}\wedge\left(V^{\alpha}\wedge\mathcal{T}^{\star\beta} + \vartheta^{\alpha}\wedge\mathcal{K}^{\star\beta}\right). \end{array}\right\} \tag{165}$$

Note that the action integral (165) is invariant under Lorentz rather than CA transformations. The Lie star $\star$ operator is defined as $\star V_\alpha = \frac{1}{3!}\eta_{\alpha\beta\mu\nu}V^\beta \wedge V^\mu \wedge V^\nu$.

The field equations are obtained from variation of I with respect to the independant gauge potentials. It is convenient to define the functional derivatives

$$\frac{\delta\mathcal{L}_{\text{gauge}}}{\delta V^\alpha} := -\overset{\text{GL}}{\nabla} N_\alpha + \overset{\text{V}}{\mathfrak{T}}_\alpha,$$
$$\frac{\delta\mathcal{L}_{\text{gauge}}}{\delta \vartheta^\alpha} := -\overset{\text{GL}}{\nabla} M_\alpha + \overset{\vartheta}{\mathfrak{T}}_\alpha, \tag{166}$$
$$\mathfrak{Z}_\alpha^{\ \beta} := \frac{\delta\mathcal{L}_{\text{gauge}}}{\delta\widehat{\Gamma}\ _\beta^{\alpha}} = -\ ^\dagger\overset{\text{GL}}{\nabla}\widehat{M}\ _\alpha^{\ \beta} + \widehat{E}\ _\alpha^{\ \beta}.$$

where

$$\widehat{M}\ _\beta^{\ \alpha} := -\frac{\partial\mathcal{L}_{\text{gauge}}}{\partial\widehat{R}\ _\alpha^{\ \beta}},\ \widehat{E}\ _\alpha^{\ \beta} := \frac{\partial\mathcal{L}_{\text{gauge}}}{\partial\widehat{\Gamma}\ _\beta^{\ \alpha}},\ \overset{\text{V}}{\mathfrak{T}}_\alpha := \frac{\partial\mathcal{L}_{\text{gauge}}}{\partial V^\alpha},\ \overset{\vartheta}{\mathfrak{T}}_\alpha := \frac{\partial\mathcal{L}_{\text{gauge}}}{\partial\vartheta^\alpha},\ \Theta := \frac{\partial\mathcal{L}_{\text{gauge}}}{\partial\Phi}. \tag{167}$$

The gauge field momenta are defined by

$$N_\alpha := -\frac{\partial\mathcal{L}_{\text{gauge}}}{\partial\mathcal{T}^\alpha},\ M_\alpha := -\frac{\partial\mathcal{L}_{\text{gauge}}}{\partial\mathcal{K}^\alpha},\ \Xi := -\frac{\partial\mathcal{L}_{\text{gauge}}}{\partial\mathcal{Z}},$$
$$\widehat{M}_{[\alpha\beta]} := N_{\alpha\beta} = -o_{[\alpha|\gamma}\frac{\partial\mathcal{L}_{\text{gauge}}}{\partial\mathcal{R}_\gamma^{\ |\beta]}},\ \widehat{M}_{(\alpha\beta)} := M_{\alpha\beta} = -2o_{(\alpha|\gamma}\frac{\partial\mathcal{L}_{\text{gauge}}}{\partial\mathfrak{R}_\gamma^{\ |\beta)}}. \tag{168}$$

Furthermore, the shear (gauge field deformation) and hypermomentum current forms are given by

$$\widehat{E}_{(\alpha\beta)} := U_{\alpha\beta} = -V_{(\alpha} \wedge \left(M_{\beta)} + N_{\beta)}\right) - M_{\alpha\beta},\ \widehat{E}_{[\alpha\beta]} := E_{\alpha\beta} = -V_{[\alpha} \wedge \left(M_{\beta]} + N_{\beta]}\right), \tag{169}$$

The analogue of the Einstein equations read

$$G_\alpha + \Lambda\widehat{\eta}_\alpha +\ ^\dagger\overset{\text{GL}}{\nabla}\mathcal{T}_{\star\alpha} + \overset{\text{V}}{\mathfrak{T}}_\alpha = 0, \tag{170}$$

with Einstein-like three-form

$$G_\alpha = \left(\mathcal{R}^{\beta\gamma} + \Upsilon^{[\beta|}{}_\rho \wedge \Upsilon^{|\gamma]\rho}\right) \wedge \left(\eta_{\beta\gamma\alpha} + \star\left[B_{\beta\gamma} \wedge \vartheta_\alpha\right]\right), \tag{171}$$

coupling constant Λ and mixed three-form $\widehat{\eta}_\alpha = \eta_\alpha + \star(\vartheta_\alpha \wedge V_\beta) \wedge V^\beta$. Observe that G_α includes symmetric GL_4 (Υ) as well as special conformal (ϑ) contributions. The gauge field 3-form $\overset{\text{V}}{\mathfrak{T}}_\alpha$ is given by

$$\begin{aligned}\overset{\text{V}}{\mathfrak{T}}_\alpha \ &= \ \langle\mathcal{L}_{\text{gauge}}|e_\alpha\rangle + \langle\mathcal{Z}|e_\alpha\rangle \wedge \Xi + \left\langle\mathcal{T}^\beta|e_\alpha\right\rangle \wedge N_\beta + \\ &+ \left\langle\mathcal{K}^\beta|e_\alpha\right\rangle \wedge M_\beta + \left\langle\mathcal{R}_\gamma^{\ \beta}|e_\alpha\right\rangle \wedge N^\gamma_{\ \beta} + \frac{1}{2}\left\langle\mathfrak{R}_\gamma^{\ \beta}|e_\alpha\right\rangle M^\gamma_{\ \beta},\end{aligned} \tag{172}$$

We remark that to interpret (171) as the gravitational field equation analogous to the Einstein equations, we must transform from the Lie algebra index α to the spacetime basis index k by contracting over the former (α) with the CA tetrads e_k^α.

$$\begin{aligned}\overset{\text{V}}{\mathfrak{T}}_\alpha \ &= \ \mathfrak{T}_\alpha\left[\mathcal{T}\right] + \mathfrak{T}_\alpha\left[\mathcal{K}\right] + \mathfrak{T}_\alpha\left[\mathcal{R}\right] + \mathfrak{T}_\alpha\left[Z\right] - \left\langle\mathcal{T}^\beta|e_\alpha\right\rangle \wedge N_\beta - \left\langle\mathcal{K}^\beta|e_\alpha\right\rangle \wedge M_\beta + \\ &- \left\langle\mathcal{R}_\gamma^{\ \beta}|e_\alpha\right\rangle \wedge N^\gamma_{\ \beta} - \langle\mathcal{Z}|e_\alpha\rangle \wedge \Xi + \Psi_{\star\alpha\beta} \wedge \vartheta^\beta + \langle\Sigma_{\star\gamma\beta}|e_\alpha\rangle \wedge \widehat{R}\ ^{\alpha\beta} + \\ &+ \left\langle\Upsilon^{\gamma\beta} \wedge \left(V_\gamma \wedge \mathcal{T}_{\star\beta} + \vartheta_\gamma \wedge \mathcal{K}_{\star\beta}\right)|e_\alpha\right\rangle + \Sigma_{\star\gamma\beta} \wedge \left\langle\widehat{R}\ ^{\gamma\beta}|e_\alpha\right\rangle + \\ &\mathcal{B}_{\star\gamma\beta} \wedge \left\langle\mathcal{B}^{\gamma\beta}|e_\alpha\right\rangle + \langle\mathcal{B}_{\star\gamma\beta}|e_\alpha\rangle \wedge \mathcal{B}^{\gamma\beta} + \langle\Psi_{\star\gamma\beta}|e_\alpha\rangle \wedge \Psi^{\gamma\beta}\end{aligned} \tag{173}$$

respectively, with

$$\begin{aligned}
\mathfrak{T}_\alpha\left[\mathcal{R}\right] &= \tfrac{1}{2}a_1\left(\mathcal{R}_{\rho\gamma}\wedge\langle\mathcal{R}^{\star\rho\gamma}|e_\alpha\rangle-\langle\mathcal{R}_{\rho\gamma}|e_\alpha\rangle\wedge\mathcal{R}^{\star\rho\gamma}\right),\\
\mathfrak{T}_\alpha\left[\mathcal{T}\right] &= \tfrac{1}{2}a_2\left(\mathcal{T}_{\gamma}\wedge\langle\mathcal{T}^{\star\gamma}|e_\alpha\rangle-\langle\mathcal{T}_{\gamma}|e_\alpha\rangle\wedge\mathcal{T}^{\star\gamma}\right),\\
\mathfrak{T}_\alpha\left[\mathcal{K}\right] &= \tfrac{1}{2}a_3\left(\mathcal{K}_{\gamma}\wedge\langle\mathcal{K}^{\star\gamma}|e_\alpha\rangle-\langle\mathcal{K}_{\gamma}|e_\alpha\rangle\wedge\mathcal{K}^{\star\gamma}\right),\\
\mathfrak{T}_\alpha\left[Z\right] &= \tfrac{1}{2}a_4\left(d\Phi\wedge\langle\star d\Phi|e_\alpha\rangle-\langle d\Phi|e_\alpha\rangle\wedge\star d\Phi\right).
\end{aligned}\tag{174}$$

From the variation of I with respect to ϑ^α we get

$$\mathfrak{G}_\alpha+\Lambda\widehat{\omega}_\alpha+\,{}^{\dagger}\overset{\mathrm{GL}}{\nabla}\mathcal{K}_{\star\alpha}+\overset{\vartheta}{\mathfrak{T}}_\alpha=0,\tag{175}$$

where in analogy to (171) we have

$$\mathfrak{G}_\alpha=h^\alpha_i\left(\mathcal{R}^{\beta\gamma}+\Upsilon^{[\beta|}{}_\rho\wedge\Upsilon^{|\gamma]\rho}\right)\wedge\left(\omega_{\beta\gamma\alpha}+\star\left[\mathcal{B}_{\beta\gamma}\wedge V_\alpha\right]\right),\tag{176}$$

where $\widehat{\omega}_\alpha=\omega_\alpha+\star\left(\vartheta_\alpha\wedge V_\beta\right)\wedge\vartheta^\beta$. The quantity $\overset{\vartheta}{\mathfrak{T}}_i=h^\alpha_i\overset{\vartheta}{\mathfrak{T}}_\alpha$ is similar to (172) but with the algebra basis e_α replaced by h_α and the CA tetrad components $e^\alpha_{\ i}$ replaced by $h^\alpha_{\ i}$. The two gravitational field equations (171) and (176) are $P-\Delta$ symmetric. We may say that they exhibit $P-\Delta$ duality symmetry invariance.

From the variational equation for $\overset{\circ}{\Gamma}{}_\alpha^{\ \beta}$ we obtain the CA gravitational analogue of the Yang-Mills-torsion type field equation,

$$\overset{\circ}{\nabla}\star\mathcal{R}_\alpha^{\ \beta}+\overset{\circ}{\nabla}\star\Sigma_\alpha^{\ \beta}+\left(V^\beta\wedge\mathcal{T}_{\star\alpha}+\vartheta^\beta\wedge\mathcal{K}_{\star\alpha}\right)=0.\tag{177}$$

Variation of I with respect to $\Upsilon_\alpha^{\ \beta}$ leads to

$$\overset{\circ}{\nabla}\star\Sigma_{\alpha\beta}-\Upsilon_{(\alpha|}{}^{\gamma}\wedge\Sigma_{\star\gamma|\beta)}+V_{(\alpha}\wedge\mathcal{T}_{\star\beta)}+\vartheta_{(\alpha}\wedge\mathcal{K}_{\star\beta)}=0.\tag{178}$$

Finally, from the variational equation for Φ, the gravi-scalar field equation is given by

$$d\star d\Phi+V_\alpha\wedge\mathcal{T}^{\star\alpha}+\vartheta_\alpha\wedge\mathcal{K}^{\star\alpha}=0.\tag{179}$$

The field equations of CA gravity were obtained in this section. The analogue of the Einstein equation, obtained from variation of I with respect to the coframe V, is characterized by an Einstein-like 3-form that includes symmetric GL_4 as well as special conformal contributions. Moreover, the field equation in (171) contains a non-trivial torsion contribution. Performing a $P-\Delta$ transformation (i.e. $V\rightarrow\vartheta$, $\mathcal{T}\rightarrow\mathcal{K}$, $D\rightarrow -D$) on (171) we obtain (176). This result may also be obtained directly by varying I with respect ϑ. A mixed CA cosmological constant term arises in (171), (176)) as a consequence of the structure of the 2-form $\mathbb{R}^\alpha_{\ \beta}$.

The field equation (177) is a Yang-Mills-like equation that represents the generalization of the Gauss torsion-free equation $\nabla\star B^{\alpha\beta}=0$. In our case, we considered a mixed volume form involving both V and ϑ leading to the substitution $B^{\alpha\beta}\rightarrow\Sigma^{\alpha\beta}$. Additionally, even in the case of vanishing $T^\rho=\overset{\circ}{\nabla}V^\rho$, the CA torsion depends on the dilaton potential Φ which in general is non-vanishing. A similar argument holds for the special conformal quantity $\mathcal{K}^\rho$. Admitting the quadratic curvature term $\mathcal{R}_\alpha^\beta\wedge\star\mathcal{R}_\beta^\alpha$ in the gauge Lagrangian it becomes clear how we draw the analogy between (177) and the Gauss equation. Equation (178) follow from similar considerations as (177), the significant differences being the lack of a $\overset{\circ}{\nabla}\star\mathfrak{R}_\alpha^{\ \beta}$ counterpart to $\overset{\circ}{\nabla}\star\mathcal{R}_\alpha^{\ \beta}$ since $\star\mathfrak{R}_\alpha^{\ \beta}=0$. Finally, (179) involves both $\mathcal{T}^\rho$ and $\mathcal{K}^\rho$ in conjunction with a term that resembles the source-free maxwell equation with the dilaton potential playing a similar role to the electromagnetic vector potential.

X. CONCLUSION

In this paper a nonlinearly realized representation of the local CA group was determined. It was found that the nonlinear Lorentz transformation law contains contributions from the linear Lorentz parameter as well as conformal and shear contributions via the nonlinear 4-boosts and symmetric GL_4 parameters. We identified the pullback of

the nonlinear translational connection coefficient to M as a spacetime coframe. In this way, the frame fields of the theory are obtained from the (nonlinear) gauge prescription. The mixed index coframe component (tetrad) is used to convert from Lie algebra indices into spacetime indices. The spacetime metric is a secondary object constructed from the constant H group metric and the tetrads. The gauge fields $\overset{\circ}{\Gamma}{}^{\alpha\beta}$ are the analogues of the Christoffel connection coefficients of GR and serve as the gravitational gauge potentials used to define covariant derivative operators. The gauge fields ϑ, Φ, and Υ encode information regarding special conformal, dilatonic and deformational degrees of freedom of the bundle manifold. The spacetime geometry is therefore determined by gauge field interactions.

The bundle curvature and Bianchi identities were determined. The gauge Lagrangian density was modeled after the available boundary topological invariants. As a consequence of this approach, no mixed field strength terms involving different components of the total curvature arose in the action. The analogue of the Einstein equations contains a non-trivial torsion contribution. The Einstein-like three-form includes symmetric GL_4 as well as special conformal contributions. A mixed translational-conformal cosmological constant term arises due to the structure of the generalized curvature of the manifold. We also obtain a Yang-Mills-like equation that represents the generalization of the Gauss torsion-free equation. Variation of I with respect to $\Upsilon_\alpha^{\ \beta}$ leads to a constraint equation relating the GL_4 deformation gauge field to the translational and special conformal field strengths. The gravi-scalar field equation has non-vanishing translational and special conformal contributions.

XI. APPENDIX

A. Maurer-Cartan 1-forms

For the case of matrix groups, the left invariant vector (operator) belonging to the tangent space $\mathbb{T}(\mathbb{P})$ is defined by [33],

$$\widehat{\mathfrak{L}}_A = u_M^{\ \ L} \rho\left(\mathbf{G}_A\right)_L^{\ \ N} \frac{\partial}{\partial u_M^{\ \ N}}. \tag{180}$$

with $(p\widetilde{g}_\lambda)_M^{\ \ N} = u_M^{\ \ Q}\mathcal{D}_Q^{\ \ N}$, and $\mathcal{D}_Q^{\ \ N}$ is the adjoint representation matrix [27] for the Lie algebra basis $\mathbf{G}_A$. Here u is the parameterization matrix of elements $\widetilde{g}$. For instance, if $\widetilde{g} = \exp(\lambda^A_{\ B} G^B_{\ A})$, then $u^A_{\ B} := \exp(\lambda^A_{\ B})$. In terms of $\mathbf{G}_A$ we define the canonical $\mathfrak{g}$-valued one-form $\Theta = g^{-1}dg = \Theta^A \mathbf{G}_A$ ($g \in G$) on $\mathbb{P}$, inheriting the left invariance of $\mathbf{G}_A$ in terms of which it is defined, namely $L_g^* \Theta|_{gp} = \Theta|_p$. The components of Θ read

$$\Theta^A = -\frac{1}{2}\left(\gamma^{-1}\right)^{AB} \rho\left(\mathbf{G}_B\right)_M^{\ \ N} \left(u^{-1}\right)_N^{\ \ L} du_L^{\ \ M}, \tag{181}$$

where $\left(\gamma^{-1}\right)^{AB}$ is the inverse of the Cartan-Killing metric γ_{AB} whose anholonomic components are given in terms of $\mathbf{G}_A$ as [33],

$$\gamma_{AB} = -2tr\left(\mathbf{G}_A \mathbf{G}_B\right) = -2f_{AM}^{\ \ \ \ L} f_{BL}^{\ \ \ \ M}. \tag{182}$$

They satisfy

$$\gamma_{AB} = \mathcal{D}_A^{\ \ C}\mathcal{D}_B^{\ \ D}\gamma_{CD}. \tag{183}$$

The basis $\widehat{\mathfrak{L}}_A$ and one-form Θ satisfy the duality and left invariance conditions, $\left\langle \Theta | \widehat{\mathfrak{L}}_A \right\rangle = \mathbf{G}_A$ and $L_{g*} : L_{A|p} \rightarrow L_{A|gp}$. The right invariant basis vector operators are given by

$$\widehat{\mathfrak{R}}_A := \rho\left(\mathbf{G}_A\right)_M^{\ \ L} u_L^{\ \ N} \frac{\partial}{\partial u_M^{\ \ N}}, \tag{184}$$

while the canonical right invariant $\mathfrak{g}$-valued one-form $\overline{\Theta} = (dg)\, g^{-1} = \overline{\Theta}^A \mathbf{G}_A$, where

$$\overline{\Theta}^A = -\frac{1}{2}\left(\gamma^{-1}\right)^{AB} \ \rho\left(\mathbf{G}_B\right)_M^{\ \ N} du_N^{\ \ L} \left(u^{-1}\right)_L^{\ \ M} \tag{185}$$

satisfies $\left\langle \overline{\Theta} | \widehat{\mathfrak{R}}_A \right\rangle = \mathbf{G}_A$. We obtain $\Theta^{-1}\mathbf{G}_A\Theta = \mathcal{D}_A^{\ \ B}\mathbf{G}_B$, where the matrix $\mathcal{D}_A^B$ is given by

$$\mathcal{D}_A^B = \widehat{\mathfrak{L}}_A \left(\widehat{\mathfrak{R}}^{\ -1}\right)^B. \tag{186}$$

Rewriting $\mathbf{G}_A\Theta = \mathcal{D}_A^{\ B}\Theta\mathbf{G}_B$, differentiating with respect to $\widetilde{g}_\lambda$ and taking the limit $g = (id)_G$, we arrive at the commutation relations [27]:

$$\left[\widehat{\mathfrak{L}}_A,\, \widehat{\mathfrak{L}}_B\right] = f_{AB}^{\quad\ C}\widehat{\mathfrak{L}}_C, \quad \left[\widehat{\mathfrak{R}}_A,\, \widehat{\mathfrak{R}}_B\right] = -f_{AB}^{\quad\ C}\widehat{\mathfrak{R}}_C, \ \left[\widehat{\mathfrak{R}}_A,\, \widehat{\mathfrak{L}}_B\right] = 0. \tag{187}$$

With the aid of the BCH formula, we determine the explicit form of the adjoint representation of the Lie algebra basis elements $ad\left(\widetilde{g}^{-1}\right)\mathbf{G}_A = \mathcal{D}_A^{\ B}\mathbf{G}_B$,

$$\mathcal{D}_A^{\ B} = \left[e^{\lambda^M \rho(\mathbf{G}_M)}\right]_A^{\ \ B} = \delta_A^B - \lambda^C f_{CA}^{\quad\ B} + \frac{1}{2!}\lambda^C f_{CA}^{\quad\ M}\lambda^D f_{DM}^{\quad\ B} - \cdots, \tag{188}$$

where [33] use was made of $\left[\rho\left(\mathbf{G}_A\right)\right]_B^C = -f_{AB}^{\quad\ C}$.

B. Baker-Campbell-Hausdorff Formulas

In the following we make extensive use of the BCH formulas

$$\begin{aligned} e^{-A}Be^{A} &= B - \tfrac{1}{1!}\left[A, B\right] + \tfrac{1}{2!}\left[A,\left[A, B\right]\right] - \cdots, \\ e^{-\chi A}de^{\chi A} &= d\chi A - \tfrac{1}{2!}\left[\chi A, d\chi A\right] + \tfrac{1}{3!}\left[\chi A,\left[\chi A, d\chi A\right]\right] - \cdots, \\ e^{i\left(h^{\mu\nu}+\delta h^{\mu\nu}\right)\ {}^{\dagger}\mathbf{S}_{\mu\nu}} &= e^{ih^{\mu\nu}\ {}^{\dagger}\mathbf{S}_{\mu\nu}}\left[1 + ie^{-h^{\alpha}_{\ \gamma}}\delta e^{h^{\gamma\beta}}\left({}^{\dagger}\mathbf{S}_{\alpha\beta} + \mathbf{L}_{\alpha\beta}\right)\right], \\ e^{i(\phi+\delta\phi)\mathbf{D}} &= e^{i\phi\mathbf{D}}\left[1 + ie^{-h^{\alpha}_{\ \beta}}\delta e^{h^{\beta}_{\ \alpha}}\mathbf{D}\right], \end{aligned} \tag{189}$$

and [70]

$$\begin{aligned} e^{i\xi^\alpha \mathbf{P}_\alpha}\omega_\alpha^{\ \beta}\mathbf{\Lambda}^\alpha_{\ \beta}e^{-i\xi^\alpha\mathbf{P}_\alpha} &= \omega_\alpha^{\ \beta}\mathbf{\Lambda}^\alpha_{\ \beta} + \omega_\alpha^{\ \beta}\xi^\alpha\mathbf{P}_\beta, \\ e^{i\Delta^{\mu\nu}\mathbf{\Lambda}_{\mu\nu}}\kappa_\alpha^{\ \beta}\mathbf{\Lambda}^\alpha_{\ \beta}e^{-i\Delta^{\mu\mu}\mathbf{\Lambda}_{\mu\nu}} &= e^{\Delta_\alpha^{\ \mu}}\kappa_\mu^{\ \nu}e^{-\Delta_\nu^{\ \beta}}\mathbf{\Lambda}^\alpha_{\ \beta}, \\ e^{ih^{\mu\nu}\mathbf{S}_{\mu\nu}}\tau^{\alpha\beta}\mathbf{L}_{\alpha\beta}e^{-ih^{\mu\nu}\mathbf{S}_{\mu\nu}} &= e^{h^{\alpha}_{\ \mu}}\tau^{\mu\nu}e^{-h_\nu^{\ \beta}}\mathbf{\Lambda}_{\alpha\beta}, \\ e^{ih^{\mu\nu}\mathbf{S}_{\mu\nu}}\sigma^{\alpha\beta}\ {}^{\dagger}\mathbf{S}_{\alpha\beta}e^{-ih^{\mu\nu}\mathbf{S}_{\mu\nu}} &= e^{h^{\alpha}_{\ \mu}}\sigma^{\mu\nu}e^{-h_\nu^{\ \beta}}\ {}^{\dagger}\mathbf{\Lambda}_{\alpha\beta}, \end{aligned} \tag{190}$$

with $\omega_\alpha^{\ \beta}\ {}^{\dagger}\mathbf{\Lambda}_\beta^{\ \alpha} = \alpha_\alpha^{\ \beta}\ {}^{\dagger}\mathbf{S}^\alpha_{\ \beta} + \beta_\alpha^{\ \beta}\mathbf{L}^\alpha_{\ \beta}$. The components of the stress forms

$$\begin{aligned} &\alpha\wedge\star\beta = \beta\wedge\star\alpha, \ \ \rho\wedge\star\sigma = \sigma\wedge\star\rho, \\ &\left\langle\left(\alpha\wedge\gamma\right)|v\right\rangle = \left\langle\alpha|\nu\right\rangle\wedge\gamma + (-1)^p\,\alpha\wedge\left\langle\gamma|\nu\right\rangle, \\ &\tfrac{\delta(\alpha\wedge\star\beta)}{\delta V} = -\delta V^c\wedge\left(\left\langle\beta|e_c\right\rangle\wedge\star\alpha - (-)^p\,\alpha\wedge\left\langle\star\beta|e_c\right\rangle\right), \\ &\tfrac{\delta(\rho\wedge\star\sigma)}{\delta\vartheta} = -\delta\vartheta^c\wedge\left(\left\langle\sigma|h_c\right\rangle\wedge\star\rho - (-)^r\,\rho\wedge\left\langle\star\sigma|h_c\right\rangle\right). \end{aligned} \tag{191}$$

In the set of equations displayed in (4.130), v is a vector, α and β are p-forms that are independent of the coframe V, while ρ and σ are r-forms that are independent of the special conformal coframe-like quantity ϑ.

Notation

$\partial_\mu = \frac{\partial}{\partial x^\mu}$: Partial derivative with respect to $\{x_\mu\}$
$\{e_\mu\}$: Set with elements e_μ
$\nabla_\mu = \partial_\mu + \Gamma_\mu$ Gauge covariant derivative operator
Γ_μ : Gauge potential 1-form
d : Exterior derivative operator
$\langle V|e\rangle$: Inner multiplication between vector e and 1-form V
$[A, B]$: Commutator of operators A and B
$\{A, B\}$: Anti-commutator of operators A and B
$\wedge$: Exterior multiplication operator
$\rtimes$: Semi-direct product
$\times$: Direct product
$\times_M$: Fibered product over manifold M
$\oplus$: Direct sum
$\otimes$: Tensor product
$A \cup B$: Union of A and B
$A \cap B$: Intersection of A and B
$\mathbb{P}(M, G; \pi)$: Fiber bundle with base space M and G-diffeomorphic fibers
$\pi_{\mathbb{P}M} : \mathbb{P} \to M$: Canonical projection map from $\mathbb{P}$ onto M
R_h, (L_h) : Right (left) group action or translation
$\widehat{\mathfrak{R}}$ $(\widehat{\mathfrak{L}})$: Right (left) invariant fundamental vector operators
Θ $(\overline{\Theta})$: Right (left) invariant Maurer-Cartan 1-form
$\circ$: Group (element) composition operator
$o_{\alpha\beta} = diag(-1, 1, 1, 1)$ or $\eta_{ij} = diag(-1, 1, 1, 1)$: Lorentz group metric
$A(4, \mathbb{R})$: Group of affine transformations on a real 4-dimensional manifold
Diff$(4, \mathbb{R})$: Group of diffeomorphisms on a real 4-dimensional manifold
$GL(4, \mathbb{R})$: Group of real 4×4 invertible matrices
$SO(4, 2)$: Special conformal group
$SO(3, 1)$: Lorentz group
$P(3, 1)$: Poincaré group
$\mathfrak{g}$: Lie algebra of group G
$g \in G$: Element g of G
$\{\mathcal{U}\} \subset M$: Set $\mathcal{U}$ is a subset of M
$\mathbf{G}$: Algebra generator of group G
$\rho(\mathbf{G})$: Representation of G-algebra
C^∞ : Infinitely differentiable (continuous)
${}^*A$: Dual of A with respect to (coordinate) basis indices
${}^\star A$: Dual of A with respect to Lie algebra indices
$\epsilon_{a_1 \ldots a_n}$ or $\varepsilon_{a_1 \ldots a_n}$: Levi-Civita totally skew tensor density
$\eta_{a_1 \ldots a_n}$: Eta basis volume n-form density
σ^* : Pullback by local section σ
L_{h*} : Differential (pushforward) map induced by L_h
$T_{(a_1 \ldots a_n)}$: Symmetrization of indices
$T_{[a_1 \ldots a_n]}$: Antisymmetrization of indices
$T(M)$: Tangent space to manifold M
$T^*(M)$: Cotangent space to M dual to $T(M)$
${}^\dagger T_{\mu\nu}$: Traceless matrix
$A^\dagger$: Hermitian adjoint of A
$f : A \to B$: Map f taking elements $\{a\} \in A$ to $\{b\} \in B$
$h : C \hookrightarrow D$: Inclusion map, where $C \subset D$

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